MMAT5360 Game Theory
2019-2020 Term 2 Home Assignment

Due: 22 April 2020 (Wednesday)

1. Solve the following matrices.

   (a) $\begin{pmatrix} 2 & 1 & 0 & 2 & -3 \\ 1 & -1 & 2 & 4 & 3 \\ 3 & 4 & 1 & -1 & -2 \end{pmatrix}$

   (b) $\begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 2 \\ 5 & -1 & -1 \end{pmatrix}$

   Answers: (a) Since the fourth column is dominated by the last column, A can be reduced to

   $$A' = \begin{pmatrix} 2 & 1 & 0 & -3 \\ 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{pmatrix}.$$

   Since the first row is dominated by the third row, A' can be reduced to

   $$A'' = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{pmatrix}.$$

   Draw the graph of

   $$\begin{cases} C_1 : v = x + 3(1-x) = 3 - 2x \\ C_2 : v = -x + 4(1-x) = 4 - 5x \\ C_3 : v = 2x + (1-x) = 1 + x \\ C_4 : v = 3x - 2(1-x) = 5x - 2 \end{cases}.$$

   The lower envelope is shown in Figure 1. Solving

   $$\begin{cases} C2 : v = 4 - 5x \\ C4 : v = 5x - 2 \end{cases},$$

   we have $v = 1$ and $x = 0.6$. Hence $v(A) = 1$ and the optimal strategy for the row player is $(0, 0.6, 0.4)$. Solving

   $$\begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} y_2 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

   we have $y_2 = y_5 = 0.5$. Therefore, the maximin strategy for the row player is $(0, 0.6, 0.4)$, the minimax strategy for the column player is (0,0.5,0,0,0.5) and the value of the game is 1.

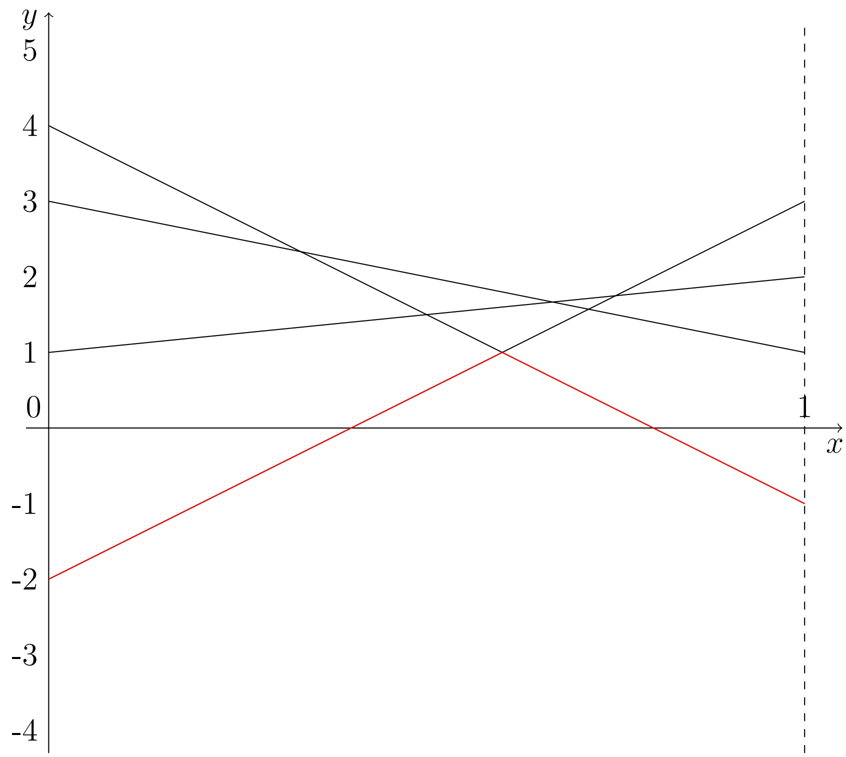


Figure 1:

(b) Add $k = 1$ to every entry to get

$$B = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 6 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{p} = (x_1, x_2, x_3)$ be a maximin strategy for the row player, $\mathbf{q} = (y_1, y_2, y_3)$ be a minimax strategy for the column player and ν_B be the value of B. Solve $\mathbf{p}B = (\nu_B, \nu_B, \nu_B)$ with $x_1 + x_2 + x_3 = 1$, we have $\mathbf{p} = (1/2, 1/3, 1/6)$ and $\nu_B = 1$. Solve $B\mathbf{q}^T = (\nu_B, \nu_B, \nu_B)^T$ with $y_1 + y_2 + y_3 = 1$, we have $\mathbf{q} = (1/6, 1/2, 1/3)$. Therefore, the value of the game is $\nu_A = \nu_B - k = 0$.

2. In a game, Ada and Bella choose one number from 2, 3 and 8 simultaneously. If the two numbers are equal, Bella pays an amount, in dollars, equal to the number to Ada. If the two numbers are different, Ada pays Bella \$1.

   (a) Write down the game matrix.

   (b) Solve the game, that is, find the maximin strategy, minimax strategy and the value of the game.

   Answer: (a) The game matrix is

$$\begin{array}{ll} Ada \backslash Bella & \begin{array}{ccc} \ \ 2 & \ \ 3 & \ \ 8 \end{array} \\ \begin{array}{l} 2 \\ 3 \\ 8 \end{array} & \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 8 \end{pmatrix} \end{array}$$

(b) Add $k = 1$ to every entry to get

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Let $\mathbf{p} = (x_1, x_2, x_3)$ be a maximin strategy for the row player, $\mathbf{q} = (y_1, y_2, y_3)$ be a minimax strategy for the column player and ν_B be the value of B. Solve $\mathbf{p}B = (\nu_B, \nu_B, \nu_B)$ with $x_1+x_2+x_3 = 1$, we have $\mathbf{p} = (12/25, 9/25, 4/25)$ and $\nu_B = 36/25$. Solve $B\mathbf{q}^T = (\nu_B, \nu_B, \nu_B)^T$ with $y_1 + y_2 + y_3 = 1$, we have $\mathbf{q} = (12/25, 9/25, 4/25)$. Therefore, the value of the game is $\nu_A = \nu_B - k = 11/25$.

3. Use simplex method to solve the zero sum game with game matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Answers: Add $k = 1$ to every entry to get

$$\begin{pmatrix} 1 & 0 & 2 \\ 4 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & 1 & 0 & 2 & 1 \\ x_2 & 4 & 3^* & 0 & 1 \\ x_3 & 0 & 1 & 2 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array} \rightarrow \begin{array}{c|ccc|c} & y_1 & x_2 & y_3 & \\ \hline x_1 & 1 & 0 & 2 & 1 \\ y_2 & 4/3 & 1/3 & 0 & 1/3 \\ x_3 & -4/3 & -1/3 & 2^* & 2/3 \\ \hline & -1/3 & -1/3 & 1 & -1/3 \end{array} \rightarrow$$

$$\rightarrow \begin{array}{c|ccc|c} & y_1 & x_2 & x_3 & \\ \hline x_1 & 7/3^* & 1/3 & -1 & 1/3 \\ y_2 & 4/3 & 1/3 & 0 & 1/3 \\ y_3 & -2/3 & -1/6 & 1/2 & 1/3 \\ \hline & 1/3 & -1/6 & -1/2 & -2/3 \end{array} \rightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & 3/7 & 1/7 & -3/7 & 1/7 \\ y_2 & -4/7 & & & 1/7 \\ y_3 & 2/7 & & & 3/7 \\ \hline & -1/7 & -3/14 & -5/14 & -5/7 \end{array}.$$

Therefore, $d = 5/7$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = (0.2, 0.3, 0.5),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (0.2, 0.2, 0.6),$$

the value of the game is $v = \frac{1}{d} - k = 2/5$.

4. Find all Nash equilibria of the bimatirx game

$$\begin{pmatrix} (4,2) & (2,3) & (3,4) \\ (3,1) & (5,5) & (1,2) \end{pmatrix}$$

Answers:

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 5 & 2 \end{pmatrix}.$$

Since the first column of B is strictly dominated by the second column of B, (A, B) can be reduced to

$$A' = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}, B' = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}.$$

Consider

$$A'\mathbf{y}^T = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3-y \\ 4y+1 \end{pmatrix},$$

we have

$$\begin{cases} 3-y > 4y+1 \ if \ 0 \le y < 0.4, \\ 3-y = 4y+1 \ if \ y = 0.4, \\ 3-y < 4y+1 \ if \ 1 \ge y > 0.4. \end{cases}$$

Thus,

$$P = \{(x,y) : (x = 0 \cap 1 \ge y \ge 0.4) \cup (0 \le x \le 1 \cap y = 0.4) \cup (x = 1 \cap 0 \le y \le 0.4)\}.$$

Consider

$$\mathbf{x}B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 5-2x & 2+2x \end{pmatrix},$$

we have

$$\begin{cases} 5-2x < 2+2x \ if \ 1 \ge x > 3/4, \\ 5-2x = 2+2x \ if \ x = 3/4, \\ 5-2x > 2+2x \ if \ 3/4 > x \ge 0. \end{cases}$$

Thus,

$$Q = \{(x,y) : (1 \ge x \ge 3/4 \cap y = 0) \cup (x = 3/4 \cap 0 \le y \le 1) \cup (3/4 \ge x \ge 0 \cap y = 1)\}.$$

By Figure 2, $P \cap Q = \{(0,1), (1,0), (3/4, 0.4)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \big((0,1), (0,1,0)\big)$, $(\mathbf{p}, \mathbf{q}) = \big((1,0), (0,0,1)\big)$, $(\mathbf{p}, \mathbf{q}) = \big((3/4, 1/4), (0, 0.4, 0.6)\big)$, and the payoff is $(5,5)$, $(3,4)$, $(2.6, 3.5)$ respectively.

5. Consider the game with bimatrix

$$(A, B) = \begin{pmatrix} (3,3) & (1,5) & (4,1) \\ (4,2) & (2,-2) & (3,2) \end{pmatrix}$$

Let ν_A and ν_{B^T} be the maximin values of A and the transpose B^T of B respectively.

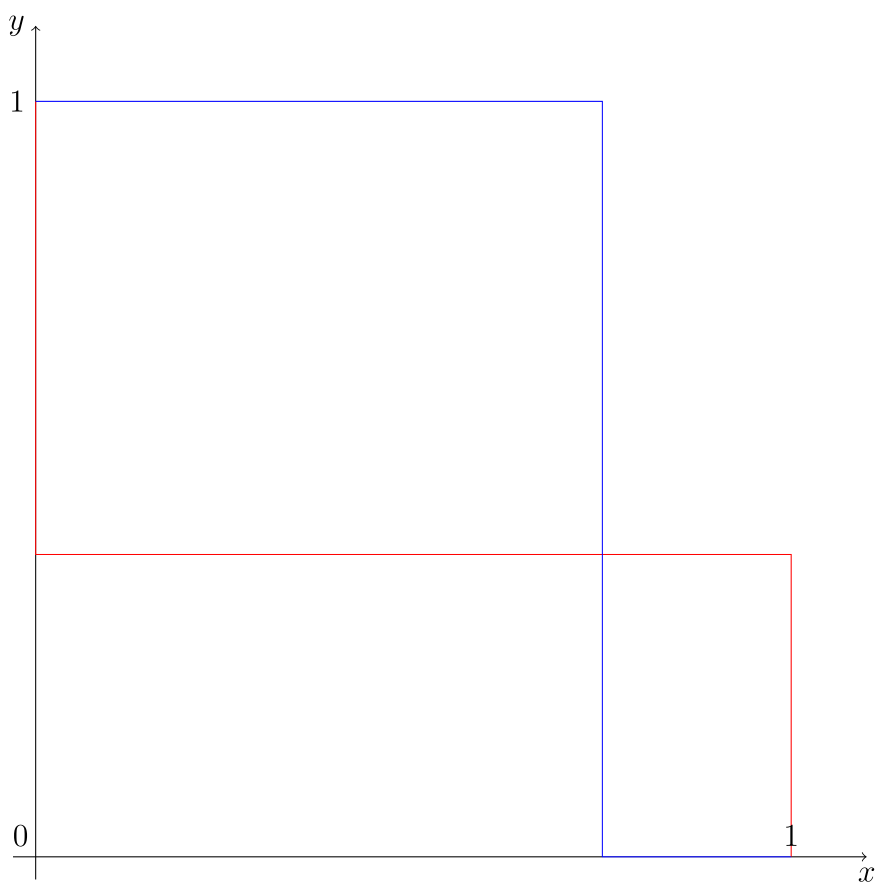


Figure 2:

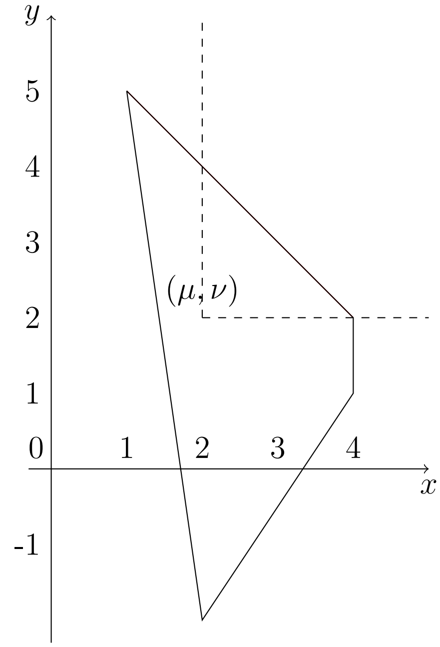


Figure 3:

(a) Find ν_A and ν_{B^T}.

(b) Sketch the cooperative region of the game.

(c) Using $(\mu,\nu) = (\nu_A, \nu_{B^T})$ as the status quo point, find the arbitration payoff pair of the game and the joint strategy to realize the arbitration.

Answers: (a)

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 4 & 2 & 3 \end{pmatrix},$$

$$B^T = \begin{pmatrix} 3 & 2 \\ 5 & -2 \\ 1 & 2 \end{pmatrix}.$$

$$(\mu,\nu) = (\nu_A, \nu B^T) = (2,2).$$

(b) See Figure 3.

(c) The equation of the line segment joining $(1,5)$ and $(4,2)$ is given by $v = 6-u$ and $g(u,v) = (u-2)(v-2) = (u-2)(4-u)$, $u \in [2,4]$, which attains its maximum at $u = 3$. Thus, the arbitration pair is $(\alpha,\beta) = (3,3)$ and the joint strategy is, for any $t \in [0,1]$,

$$P = \begin{pmatrix} 1-t & 1/3t & 0 \\ 2/3t & 0 & 0 \end{pmatrix}.$$

6. Given the game bimatrix

$$(A,B) = \begin{pmatrix} (2,5) & (8,2) \\ (4,0) & (5,4) \end{pmatrix}$$

(a) Write down the threat matrix.

(b) Find the threat strategies of the players.

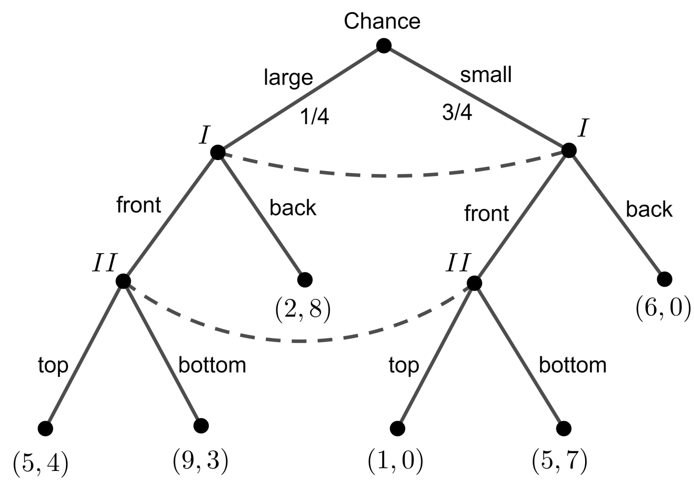
(c) Find the threat solution.

Answers: The maximum total payoff is $8+2=10$ and the threat matrix is

$$T = A - B = \begin{pmatrix} -3 & 6 \\ 4 & 1 \end{pmatrix}.$$

The threat strategies are then easily determined to be $\mathbf{p}_d = (1/4, 3/4)$ and $\mathbf{q}_d = (5/12, 7/12)$. The threat differential is $\delta = 9/4$ and the threat solution is $(\varphi_1, \varphi_2) = (\frac{10+9/4}{2}, \frac{10-9/4}{2}) = (49/8, 31/8)$.

7. Consider the game with the following game tree.



The chance of large and small are 0.25 and 0.75 respectively.

(a) Write down the strategic form (game bimatrix) of the game.

(b) Find the Nash equilibrium of the game.

Answers: (a)

$$\begin{array}{ccc} I\backslash II & top & bottom \\ front & (2,1) & (6,6) \\ back & (5,2) & (5,2) \end{array}$$

(b)

$$A = \begin{pmatrix} 2 & 6 \\ 5 & 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 6 \\ 2 & 2 \end{pmatrix}.$$

The first column of B is dominated by the second column of B but not strictly dominated.

Consider

$$A\mathbf{y}^T = \begin{pmatrix} 2 & 6 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 6-4y \\ 5 \end{pmatrix},$$

we have

$$\begin{cases} 6-4y > 5 \ if \ 0 \le y < 0.25, \\ 6-4y = 5 \ if \ y = 0.25, \\ 6-4y < 5 \ if \ 1 \ge y > 0.25. \end{cases}$$

Thus,

$P = \{(x,y) : (x = 0 \cap 1 \ge y \ge 0.25) \cup (0 \le x \le 1 \cap y = 0.25) \cup (x = 1 \cap 0 \le y \le 0.25)\}$.

Consider

$$\mathbf{x}B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2-x & 2+4x \end{pmatrix},$$

we have

$$\begin{cases} 2-x < 2+4x \ if \ 1 \ge x > 0, \\ 2-x = 2+4x \ if \ x = 0. \end{cases}$$

Thus,

$$Q = \{(x,y) : (1 \ge x > 0 \cap y = 0) \cup (x = 0 \cap 0 \le y \le 1)\}.$$

$P \cap Q = \{(x = 0 \cap 1 \ge y \ge 0.25)\} \cup \{(1,0)\}$. Therefore, the game has infinite Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \big((1,0),(0,1)\big)$, $(\mathbf{p}, \mathbf{q}) = \big((0,1),(y,1-y)\big)$ for any $y \in [0.25, 1]$, and the payoff is $(6,6)$, $(5,2)$ respectively.

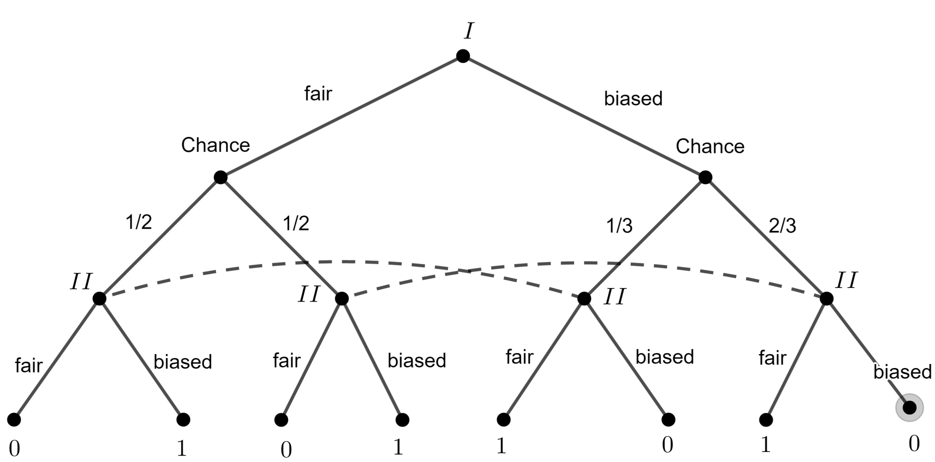
8. Player I has two coins. One is fair (probability 1/2 of heads and 1/2 of tails) and the other is biased with probability 1/3 of heads and 2/3 of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to Player II. Then Player II must guess whether Player I chose the fair or biased coin. If Player II is correct there is no payoff. If II is incorrect, she loses 1 dollar to Player I.

   (a) Draw the game tree.

   (b) Write down the strategic form (game bimatrix) of the game.

   (c) Solve the game.

Answers: (a)



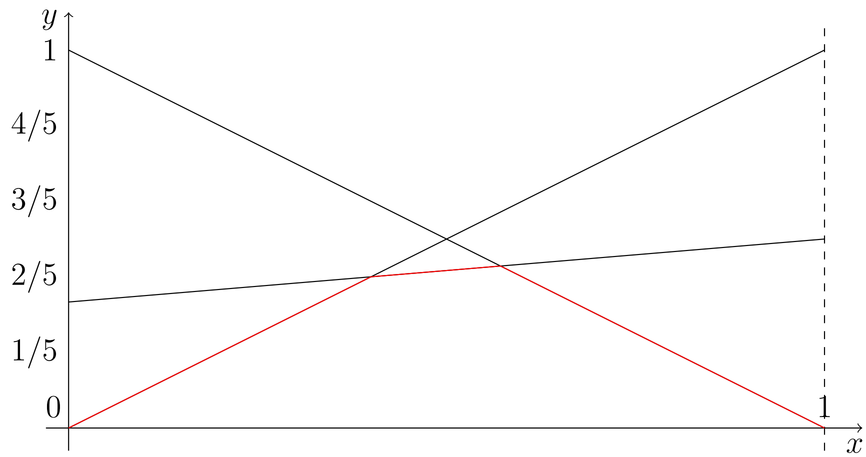


Figure 4:

(b) The game matrix is

$$\begin{array}{c|cccc} I\backslash II & FF & FB & BF & BB \\ F & 0 & 1/2 & 1/2 & 1 \\ B & 1 & 1/3 & 2/3 & 0 \end{array}$$

(c) By deleting the dominated columns, we obtain

$$A' = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 1/3 & 0 \end{pmatrix}.$$

Draw the graph of

$$\begin{cases} C_1 : v = 1 - x \\ C_2 : v = 1/6x + 1/3 \\ C_3 : v = x \end{cases}.$$

The lower envelope is shown in Figure 4. Solving

$$\begin{cases} C1 : v = 1 - x \\ C2 : v = 1/6x + 1/3 \end{cases},$$

we have $v = 3/7$ and $x = 4/7$. Hence the value of the game is $v(A) = 3/7$ and the optimal strategy for the row player is $(4/7, 3/7)$. Solving

$$\begin{pmatrix} 0 & 1/2 \\ 1 & 1/3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 3/7 \end{pmatrix},$$

we have $y_1 = 1/7$, $y_2 = 6/7$. Therefore, the maximin strategy for player I is $(4/7, 3/7)$, the minimax strategy for player II is $(1/7, 6/7, 0, 0)$ and the value of the game is $3/7$.

9. Consider a three-person game with characteristic function

$$\begin{aligned}\nu(\{1\}) &= 4\\ \nu(\{2\}) &= 2\\ \nu(\{3\}) &= 1\\ \nu(\{1,2\}) &= 12\\ \nu(\{1,3\}) &= 7\\ \nu(\{2,3\}) &= 11\\ \nu(\{1,2,3\}) &= 20\end{aligned}$$

   (a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{2,3\})$.

   (b) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

   (c) Find the Shapley value of player 1.

   Answers: (a) $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 0$ and $\mu(\{1,2,3\}) = 1$.

$$k = \frac{1}{\nu(\{1,2,3\}) - \nu(\{1\}) - \nu(\{2\}) - \nu(\{3\})} = 1/13,$$

   and we have

$$\begin{aligned}\mu(\{1,2\}) &= k(\nu(\{1,2\}) - \nu(\{1\}) - \nu(\{2\})) = 6/13,\\ \mu(\{1,3\}) &= k(\nu(\{1,3\}) - \nu(\{1\}) - \nu(\{3\})) = 2/13,\\ \mu(\{2,3\}) &= k(\nu(\{2,3\}) - \nu(\{2\}) - \nu(\{3\})) = 8/13.\end{aligned}$$

   (b) Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation, then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 4,\ x_2 \geq 2,\ x_3 \geq 1, \\ x_1 + x_2 \geq 12,\ x_1 + x_3 \geq 7,\ x_2 + x_3 \geq 11, \\ x_1 + x_2 + x_3 = 20. \end{cases} \tag{1}$$

   which is equivalent to

$$\begin{cases} 9 \geq x_1 \geq 4, \\ 13 \geq x_2 \geq 2, \\ 19 \geq x_1 + x_2 = 20 - x_3 \geq 12. \end{cases} \tag{2}$$

   $C(\nu)$ is the intersecting region of the three strip regions in Figure 5.

   (c) $\phi_1 = \frac{1}{3} \times 4 + \frac{1}{6} \times (12 - 2 + 7 - 1) + \frac{1}{3} \times (20 - 11) = 7.$

10. Players 1, 2, 3 and 4 have 40, 25, 20, and 15 votes respectively. In order to pass a certain resolution, 51 votes are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain resolution. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

    Answers: We have $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{4\}) = 0$, $\nu(\{1,2\}) = \nu(\{1,3\}) = \nu(\{1,4\}) = 1$, $\nu(\{2,3\}) = \nu(\{2,4\}) = \nu(\{3,4\}) = 0$ and $\nu(S) = 1$ for any S with $|S| \geq 3$. Thus, $\phi_1 = 1/2$, $\phi_2 = \phi_3 = \phi_4 = 1/6$.

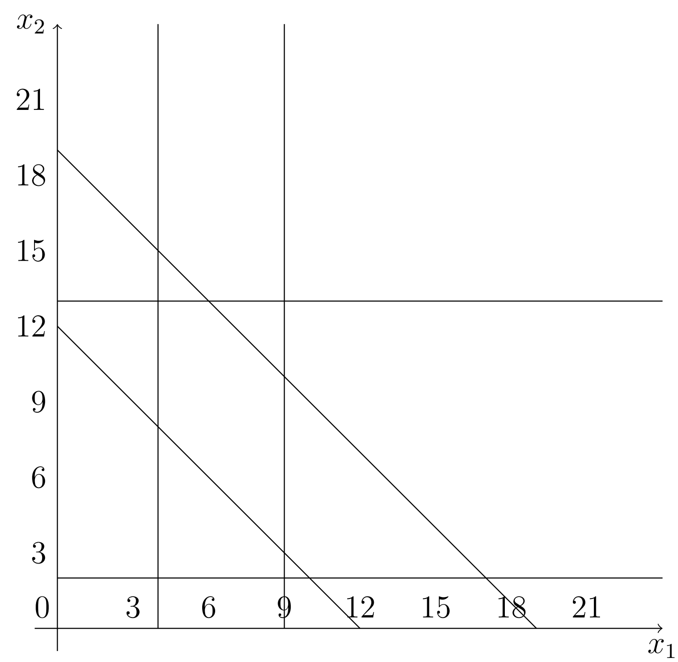


Figure 5:

11. Aaron (A), Benny (B) and Carol (C) each has to buy a book on Game Theory. The list price of the book is \$200. Alan has a discount card which allow him to buy two books for \$360, and three books for \$480. Benny has a coupon which allows him to have 20% off for the whole bill. The discount card and coupon can be used at the same time. Let $\nu(S)$ be the amount that a coalition $S \subset \{A, B, C\}$ may save by buying the books together comparing with buying them separately.

    (a) Find $\nu(\{A, B\})$, $\nu(\{B, C\})$, $\nu(\{A, C\})$ and $\nu(\{A, B, C\})$

    (b) Find $\mu(\{A, B\})$ where μ is the $(0, 1)$ reduced form of ν.

    (c) By considering the Shapley values, find the amount that each player should pay.

Answers: (a) $\nu(\{A\}) = \nu(\{B\}) = \nu(\{C\}) = 0$, $\nu(\{A, B\}) = 72$, $\nu(\{A, C\}) = 40$, $\nu(\{B, C\}) = 40$ and $\nu(\{A, B, C\}) = 176$.

(b) $\mu(\{A\}) = \mu(\{B\}) = \mu(\{C\}) = 0$ and $\mu(\{A, B, C\}) = 1$.

$$k = \frac{1}{\nu(\{A, B, C\}) - \nu(\{A\}) - \nu(\{B\}) - \nu(\{C\})} = 1/176,$$

and we have

$$\mu(\{A, B\}) = k(\nu(\{A, B\}) - \nu(\{A\}) - \nu(\{B\})) = 9/22.$$

(c)

$$\phi_A = \frac{1}{6}(72 + 40 + 2 \times (176 - 40)) = 64,$$
$$\phi_B = \frac{1}{6}(72 + 40 + 2 \times (176 - 40)) = 64,$$

$$\phi_C = \frac{1}{6}(40 + 40 + 2 \times (176 - 72)) = 48,$$

Thus, A should pay $200 - 64 = 136$ dollars, B should pay $160 - 64 = 96$ dollars and C should pay $200 - 48 = 152$ dollars.

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