Introduction to Game Theory

Lau Chi Hin
The Chinese University of Hong Kong

Prisoner's dilemma

- John and Peter have been arrested for possession of guns. The police suspects that they are going to commit a major crime.
- If no one confesses, they will both be jailed for 1 year.
- If only one confesses, he'll go free and his partner will be jailed for 5 years.
- If they both confess, they both get 3 years.

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

- If Peter confesses:

  John "confess" (3 years) better than "deny" (5 years).

- If Peter deny:

  John "confess" (0 year) better than "deny" (1 year).

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

- Thus John should confess whatever Peter does.
- Similarly, Peter should also confess.

Conclusion: Both of them should confess

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

Vickrey auction

The highest bidder wins, but the price paid is the second-highest bid.

Vickrey auction

明 報

再論以博弈論打破勾地困局

政府可考慮，如勾地者最終成功投得地皮，可讓他們享有3至5%的折扣優惠，如此建議獲接納，發展商會甘心做「出頭鳥」，搶先以高價勾地。

…其他發展商，如出價不及勾出地皮的發展商，已考慮了市場情況和財政計算，他們亦知其中一個對手享有折扣優惠，所以要打敗對手，出價只有更進取。…

也可考慮將最終成交價訂為拍賣地皮的第二最高出價。」

撰文:陸振球 (明報地產版主管)

Nobel laureates related to game theory

- 1994: Nash, Harsanyi, Selten
- 1996: Vickrey
- 2005: Aumann, Schelling
- 2007: Hurwicz, Maskin, Myerson
- 2012: Shapley, Roth
- 2014: Tirole

Price war

Two supermarkets PN and WC are engaging in a price war.

VS

Price war

- Each supermarket can choose: high price or low price.
- If both choose high price, then each will earn $4 (million).
- If both choose low price, then each will earn $2 (million).
- If they choose different strategies, then the supermarket choosing high price will earn $0 (million), while the one choosing low price will earn $5 (million).

Price war

		WC	
		Low	High
PN	Low	(2,2)	(5,0)
	High	(0,5)	(4,4)

Price war

		WC	
		Low	High
PN	Low	(2,2)	~~(5,0)~~
	High	~~(0,5)~~	~~(4,4)~~

Price war vs Prisoner dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

		WC	
		Low	High
PN	Low	(2,2)	(5,0)
	High	(0,5)	(4,4)

These are called dominant strategy equilibrium.

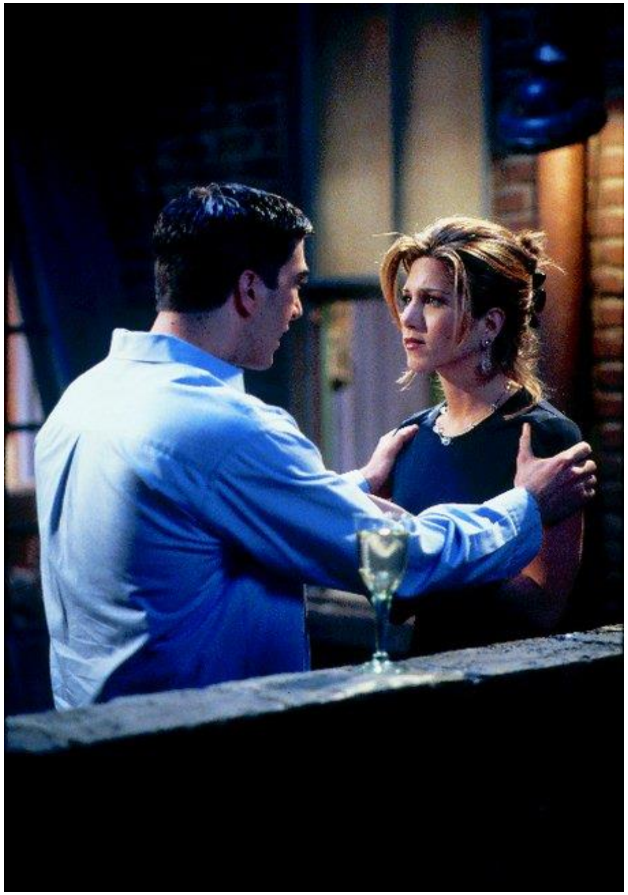
Dominant strategy equilibrium

- A strategy of a player is a dominant strategy if the player has the best return no matter how the other players play.
- If every player chooses its dominant strategy, it is called a dominant strategy equilibrium.

Dominant strategy equilibrium

- Not every game has dominant strategy equilibrium.
- A player of a game may have no dominant strategy.

Dating game



Roy and Connie would like to go out on Friday night.

Roy prefers to see football, while Connie prefers to watch drama.

However, they would rather go out together than be alone.

Dating game

		Connie	
		Football	Drama
Roy	Football	(20,5)	(0,0)
	Drama	(0,0)	(5,20)

Both Roy and Connie do not have dominant strategy. Therefore dating game does not have dominant strategy equilibrium.

Pure Nash equilibrium

- A choice of strategies of the players is a **pure Nash equilibrium** if no player can increase its gain given that *all other players do not change their strategies*.
- A dominant strategy equilibrium is always a pure Nash equilibrium.

Pure Nash equilibrium

Prisoner's dilemma

		Peter	
		Confess	Deny
John	Confess	(-3,-3)	(0,-5)
	Deny	(-5,0)	(-1,-1)

Prisoner's dilemma has a pure Nash equilibrium because it has a dominant strategy equilibrium.

Pure Nash equilibrium

Dating game

		Connie	
		Football	Drama
Roy	Football	(20,5)	(0,0)
	Drama	(0,0)	(5,20)

Dating game has no dominant strategy equilibrium but has two pure Nash equilibria.

Rock-paper-scissors

		Column player		
		Rock	Paper	Scissors
Row player	Rock	(0,0)	(-1,1)	(1,-1)
	Paper	(1,-1)	(0,0)	(-1,1)
	Scissors	(-1,1)	(1,-1)	(0,0)

Rock-paper-scissors has no pure Nash equilibrium.

Mixed strategy

Pure strategy

Using one strategy constantly.

Mixed strategy

Using varies strategies according to certain probabilities.

(Note that a pure strategy is also a mixed strategy where one of the strategies is used with probability 1 and all other strategies are used with probability 0.)

Mixed Nash equilibrium

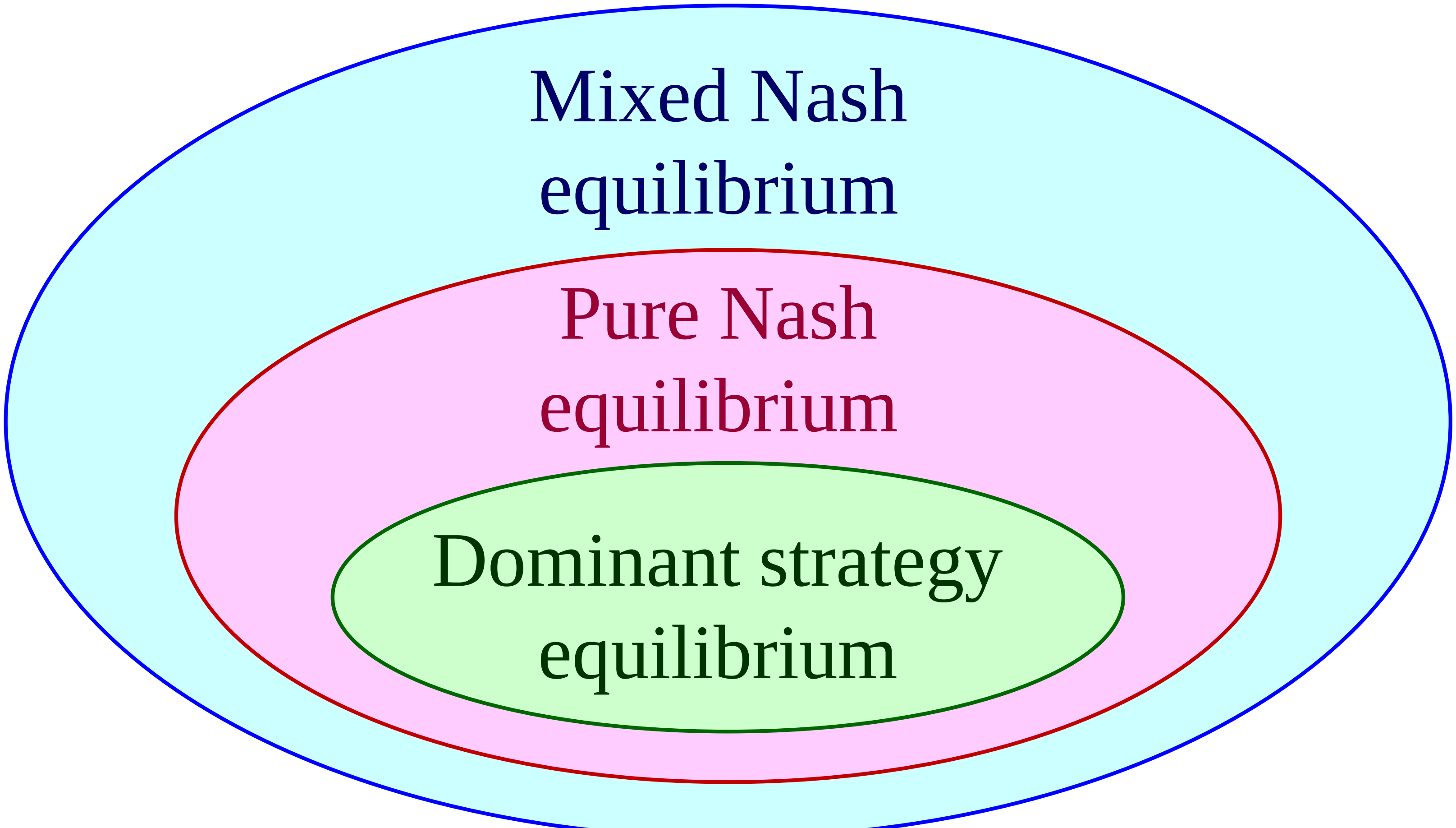
- A choice of mixed strategies of the players is called a mixed Nash equilibrium if no player has anything to gain by changing his own strategy alone while all other players do not change their strategies.
- We will simply call a mixed Nash equilibrium Nash equilibrium.

Rock-paper-scissors

		Column player		
		Rock	Paper	Scissors
Row player	Rock	(0,0)	(-1,1)	(1,-1)
	Paper	(1,-1)	(0,0)	(-1,1)
	Scissors	(-1,1)	(1,-1)	(0,0)

The mixed Nash equilibrium is both players use mixed strategy (1/3,1/3,1/3), that means all three gestures are used with the same probability 1/3.

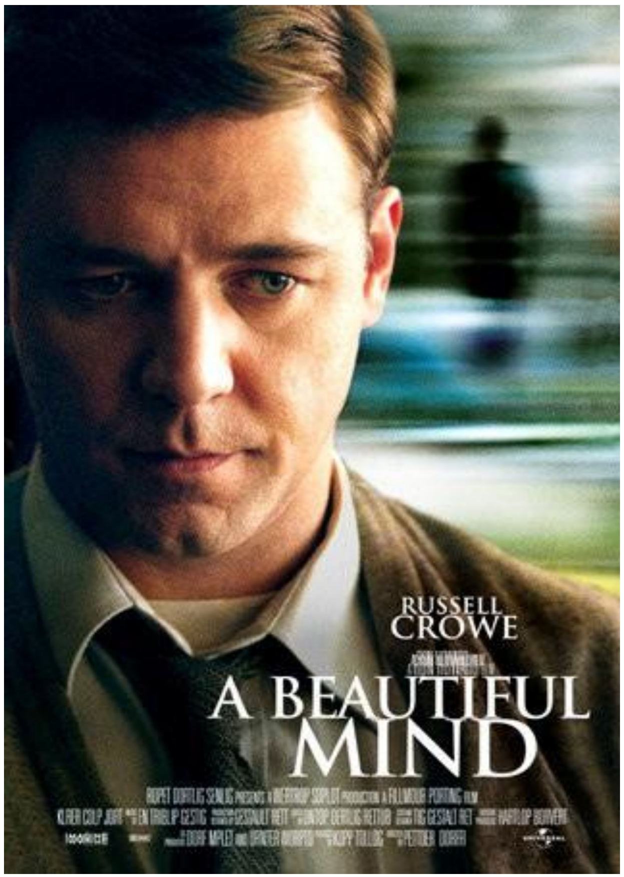
Mixed Nash equilibrium

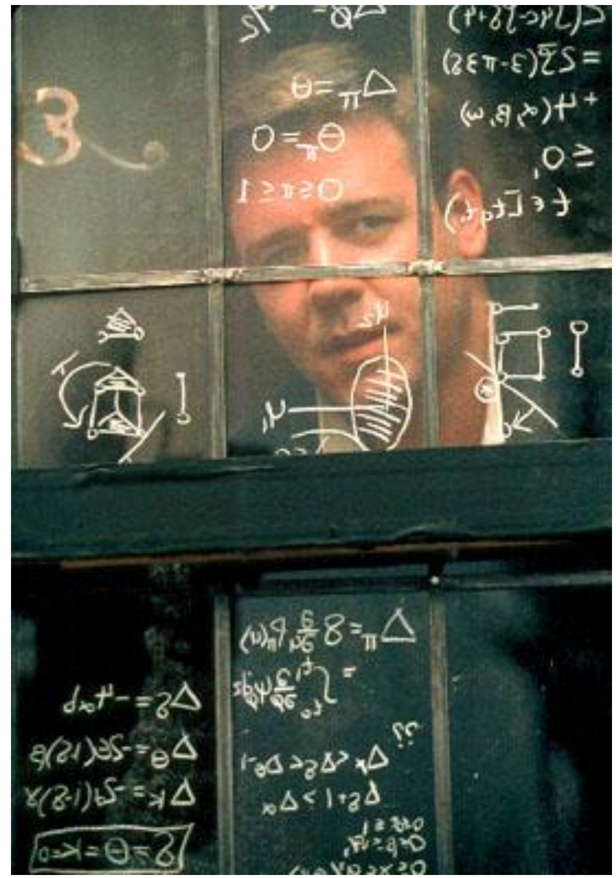


Mixed Nash equilibrium

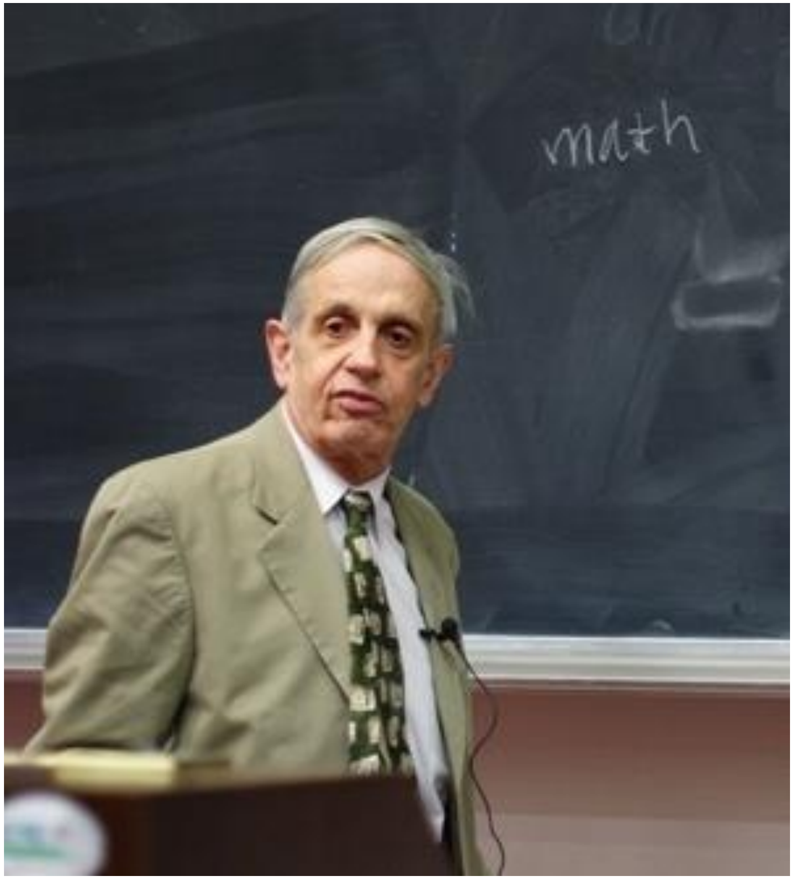
Game	Dominant strategy equilibrium	Pure Nash equilibrium	Mixed Nash equilibrium
Prisoner's dilemma	✓	✓	✓
Dating game	✗	✓	✓
Rock-paper-scissors	✗	✗	✓

A Beautiful Mind





John Nash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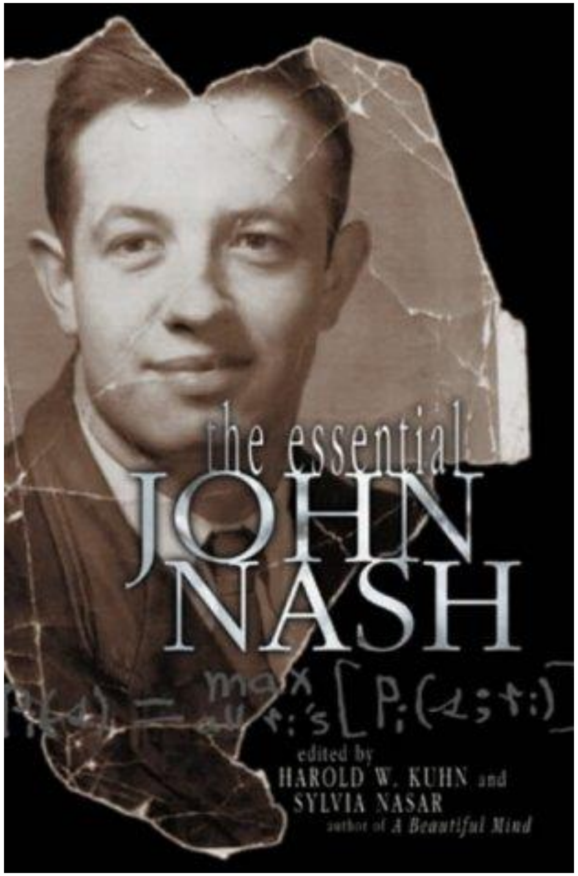
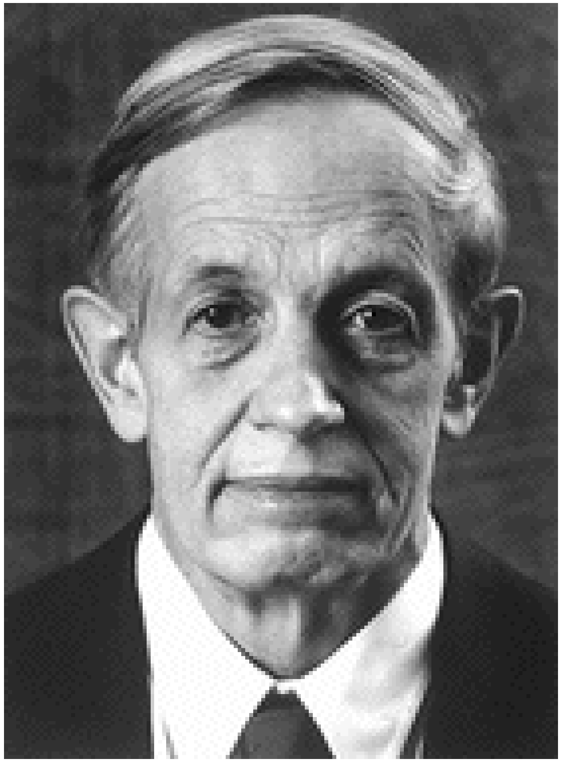

the essential
JOHN
NASH
edited by
HAROLD W. KUHN and
SYLVIA NASAR
author of A Beautiful Mind

John Nash

- Born in 1928
- Earned a PhD from Princeton in 1950 with a 28-page dissertation on non-cooperative games.

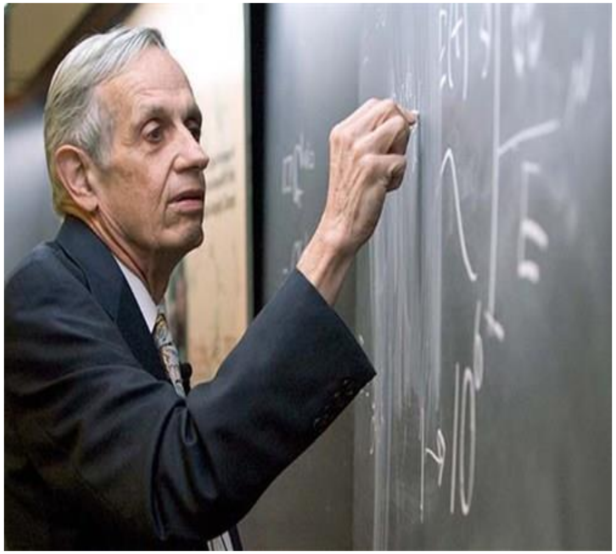


John Nash

- Married Alicia Larde, Nash’s former student in physics at MIT, in 1957

- The couple divorced in 1963 and remarried in 2001

John Nash



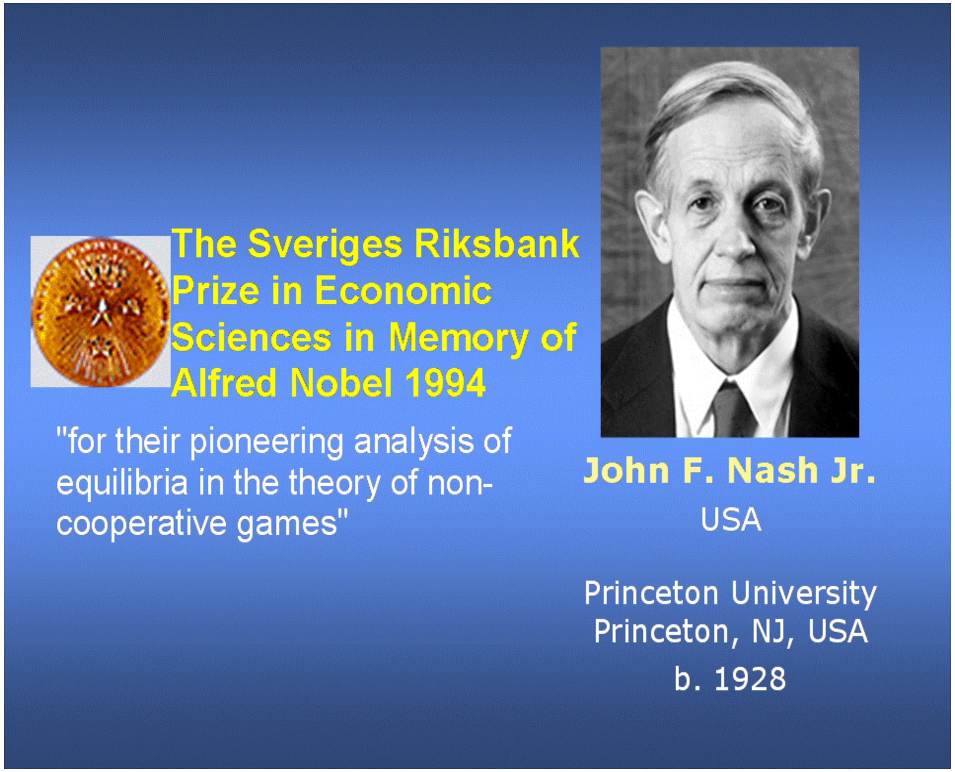
- In 1959, Nash gave a lecture at Columbia University intended to present a proof of Riemann hypothesis. However the lecture was completely incomprehensible.

John Nash

- Nash was later diagnosed as suffering from paranoid schizophrenia.
- It is a miracle that he can recover twenty years later.

John Nash

- In 1994, Nash shared the Nobel Prize in Economics with John Harsanyi and Reinhard Selten



John Nash

Notable awards

- John von Neumann Theory Prize (1978)
- Nobel Memorial Prize in Economic Sciences (1994)
- Leroy P. Steele Prize (1999)
- Abel Prize (2015)

John Nash

On May 23, 2015, Nash and his wife Alicia were killed in a collision of a taxicap. The couple were on their way home at New Jersey after visiting Norway where Nash had received the Abel Prize.

A Beautiful Mind

Nash's theory in the film

https://www.youtube.com/watch?v=bbNMTbcuitA

A Beautiful Mind

"In competition, individual ambition serves the common good."

A Beautiful Mind

"Adam Smith said the best result comes from everyone in the group doing what's best for him, right?"

"Incomplete, because the best result will come from everyone in the group doing what's the best for himself and the group.

Nash equilibrium

The example in the film is not a Nash equilibrium.

Nash embedding theorem

Any closed Riemannian n-manifold has a C^1 isometric embedding into R^{2n}.

Minimax theorem

von Neumann (Math Annalen 1928)

Minimax theorem:

For every two-person, zero-sum finite game, there exists a value v such that

- Player 1 has a mixed strategy to guarantee that his payoff is not less than v no matter how player 2 plays.
- Player 2 has a mixed strategy to guarantee that his payoff is not less than $-v$ no matter how player 1 plays.

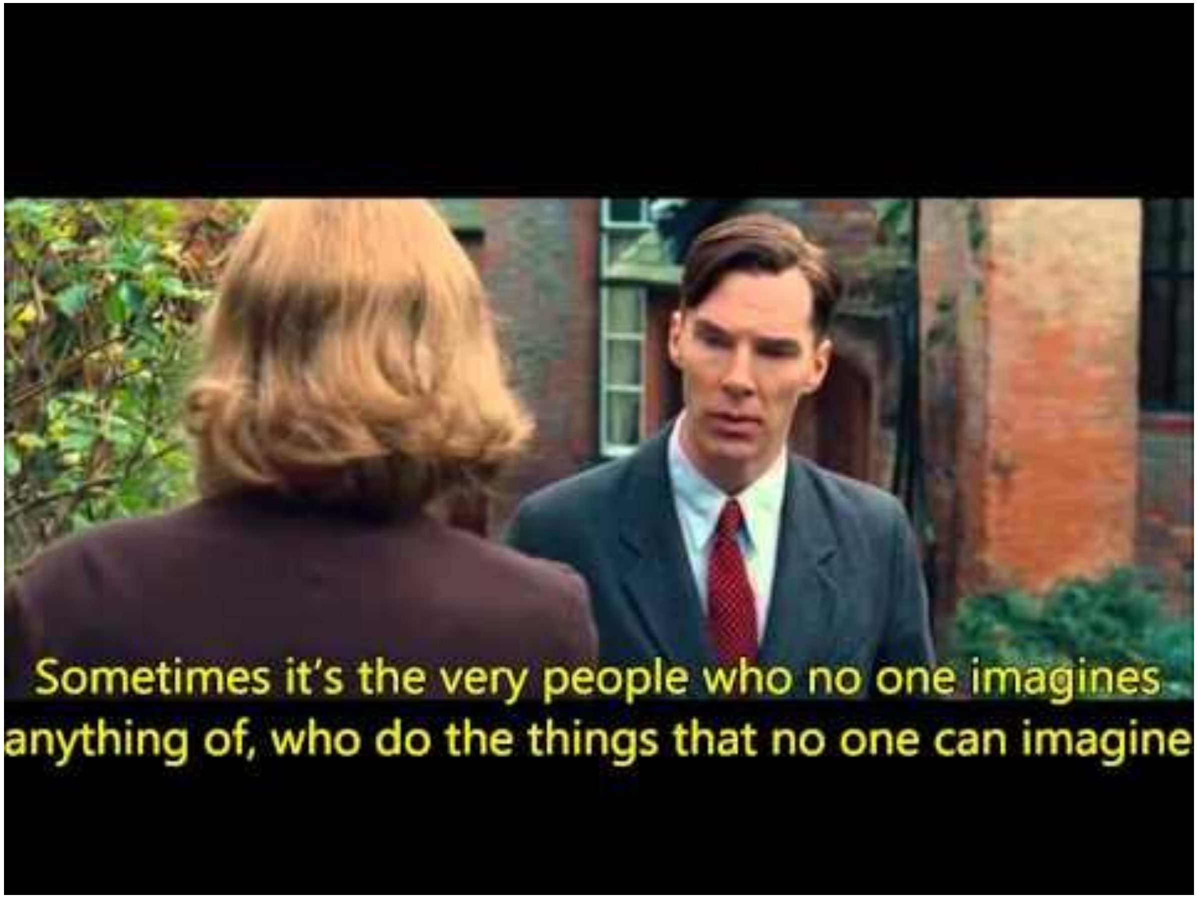
The Imitation Game

https://www.youtube.com/watch?v=Qdfp5Za0XVg

Minimax problem

The minimal number of actions it would take for us to win the war but the maximum number we can take before the Germans get suspicious.

Movie quote


Sometimes it's the very people who no one imagines
anything of, who do the things that no one can imagine

Nash's Theorem

John Nash (Annals of math 1957)
Theorem: Every finite *n*-player non-cooperative game has a mixed Nash equilibrium.

Modified rock-paper-scissors

		Column player	
		Rock	Scissor
Row player	Rock	(0,0)	(1,-1)
	Paper	(1,-1)	(-1,1)

What is the mixed Nash equilibrium?

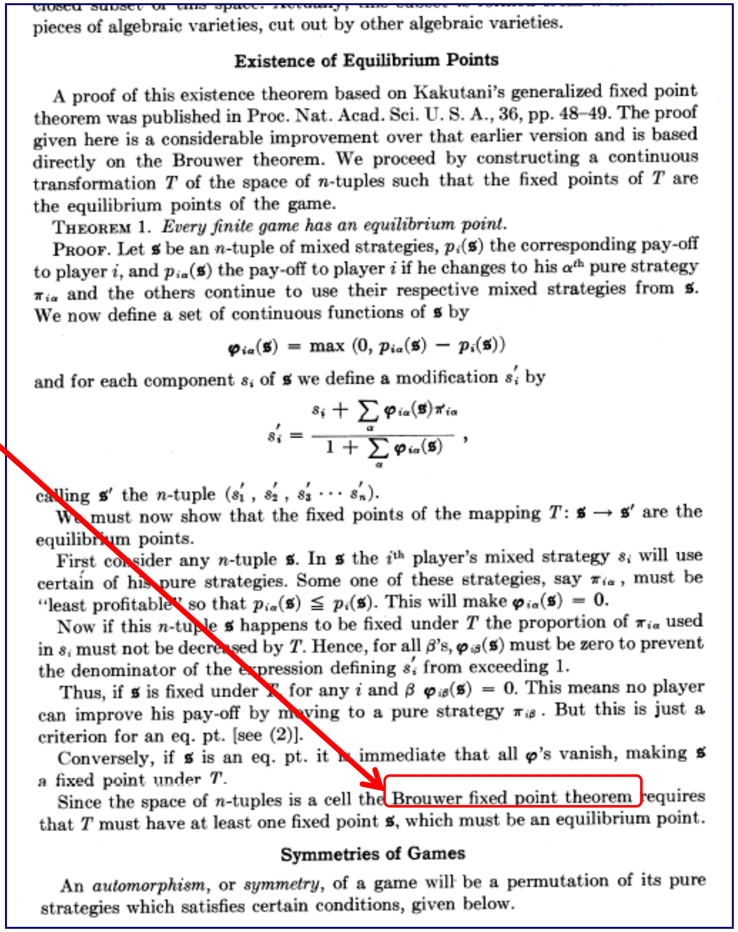
Modified rock-paper-scissors

		Column player	
		Rock	Scissor
Row player	Rock	(0,0)	(1,-1)
	Paper	(1,-1)	(-1,1)

Mixed Nash equilibrium:
Row player: (2/3,1/3)
Column player: (2/3,1/3)

Nash's Proof

Brouwer fixed-point theorem



pieces of algebraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

A proof of this existence theorem based on Kakutani's generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U. S. A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem. We proceed by constructing a continuous transformation T of the space of n-tuples such that the fixed points of T are the equilibrium points of the game.

THEOREM 1. *Every finite game has an equilibrium point.*

PROOF. Let $\mathfrak{s}$ be an n-tuple of mixed strategies, $p_i(\mathfrak{s})$ the corresponding pay-off to player i, and $p_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he changes to his α^{th} pure strategy $\pi_{i\alpha}$ and the others continue to use their respective mixed strategies from $\mathfrak{s}$. We now define a set of continuous functions of $\mathfrak{s}$ by

$$\varphi_{i\alpha}(\mathfrak{s}) = \max(0, p_{i\alpha}(\mathfrak{s}) - p_i(\mathfrak{s}))$$

and for each component s_i of $\mathfrak{s}$ we define a modification s_i' by

$$s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})},$$

calling $\mathfrak{s}'$ the n-tuple $(s_1', s_2', s_3' \cdots s_n')$.

We must now show that the fixed points of the mapping $T: \mathfrak{s} \to \mathfrak{s}'$ are the equilibrium points.

First consider any n-tuple $\mathfrak{s}$. In $\mathfrak{s}$ the i^{th} player's mixed strategy s_i will use certain of his pure strategies. Some one of these strategies, say $\pi_{i\alpha}$, must be "least profitable" so that $p_{i\alpha}(\mathfrak{s}) \leqq p_i(\mathfrak{s})$. This will make $\varphi_{i\alpha}(\mathfrak{s}) = 0$.

Now if this n-tuple $\mathfrak{s}$ happens to be fixed under T the proportion of $\pi_{i\alpha}$ used in s_i must not be decreased by T. Hence, for all β's, $\varphi_{i\beta}(\mathfrak{s})$ must be zero to prevent the denominator of the expression defining s_i' from exceeding 1.

Thus, if $\mathfrak{s}$ is fixed under T, for any i and β $\varphi_{i\beta}(\mathfrak{s}) = 0$. This means no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. But this is just a criterion for an eq. pt. [see (2)].

Conversely, if $\mathfrak{s}$ is an eq. pt. it is immediate that all φ's vanish, making $\mathfrak{s}$ a fixed point under T.

Since the space of n-tuples is a cell the Brouwer fixed point theorem requires that T must have at least one fixed point $\mathfrak{s}$, which must be an equilibrium point.

Symmetries of Games

An *automorphism*, or *symmetry*, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

Brouwer's fixed-point theorem

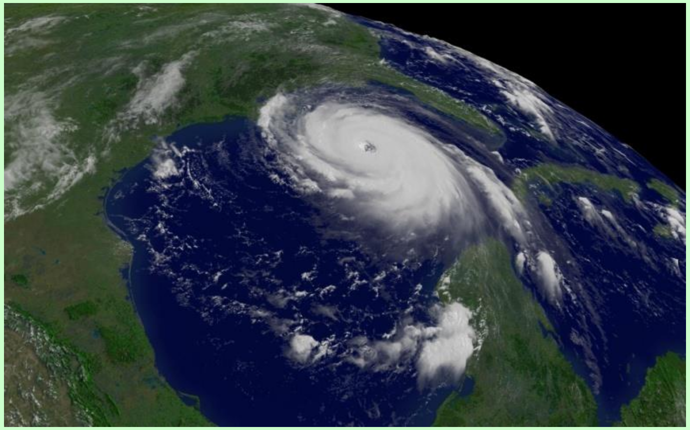
Fixed-point theorem:

Any continuous function from the *n*-dimensional closed unit ball to itself has at least one fixed-point.

Consequence of fixed-point theorem

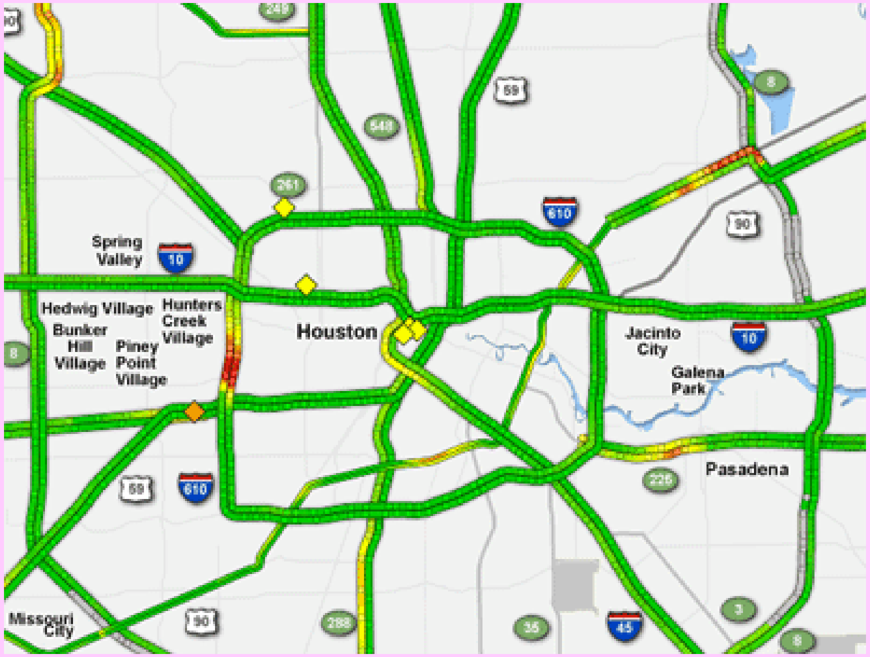
- Everybody has at least one bald spot.

- There is at least one place on earth with no wind.

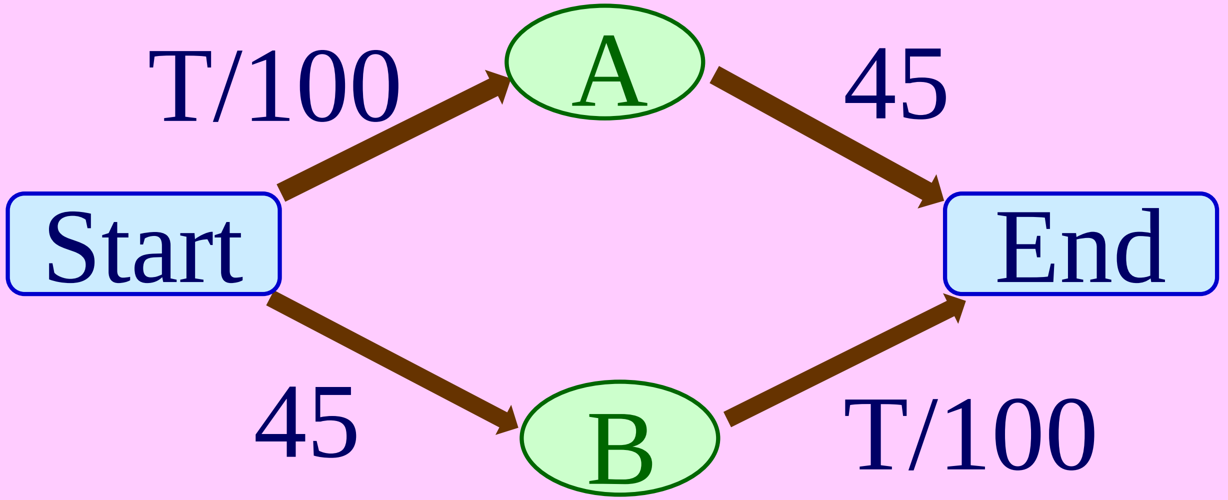


Braess paradox

Building a new road always good?



Braess paradox

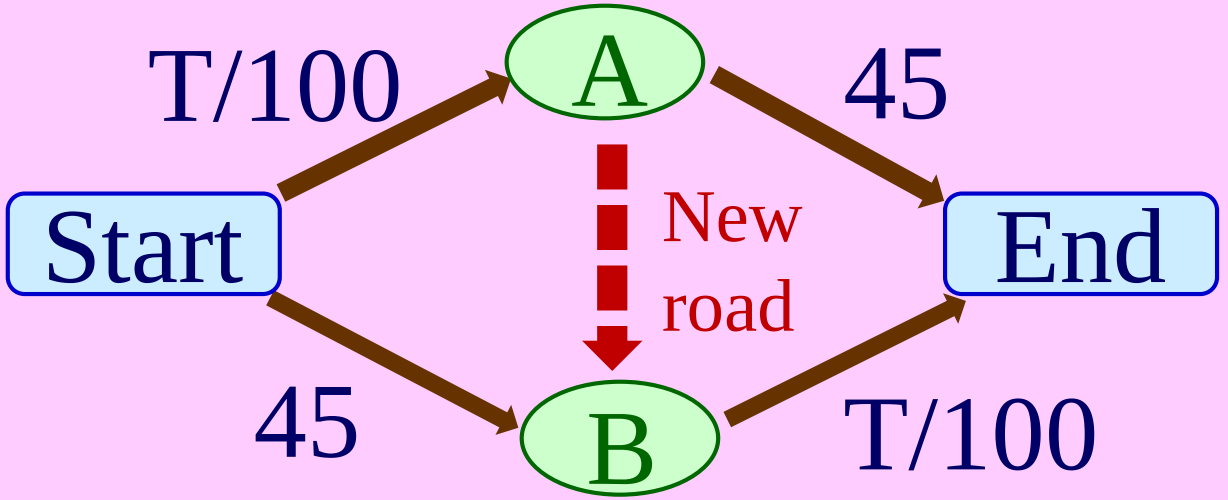


Number of vehicles:4000

Vehicles via A: 2000; Vehicles via B:2000

Expected time: 65 mins

Braess paradox

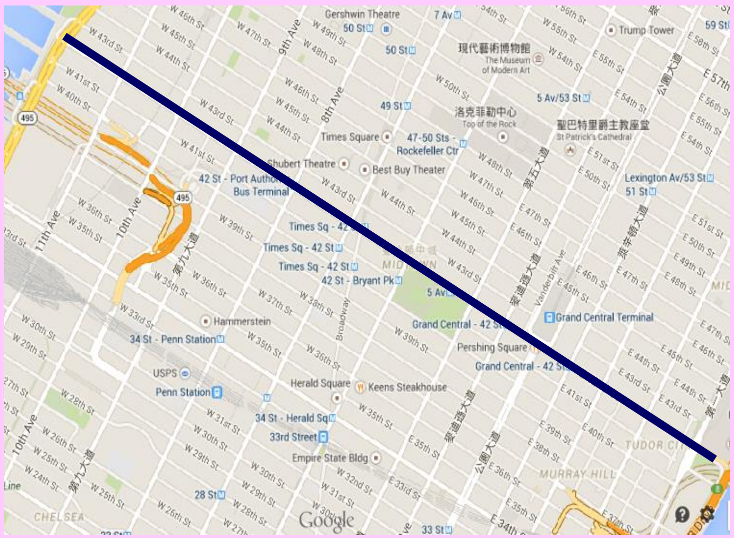


Number of vehicles:4000

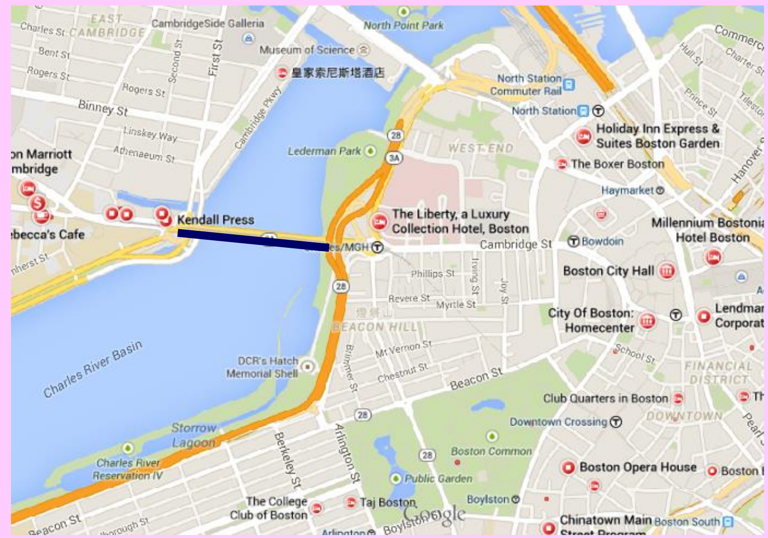
All vehicles via A and B

Expected time: 80 mins

Braess paradox in traffic network



New York City
42^{nd} Street



Boston
Main Street

Hotelling model

Hotelling model:

https://www.youtube.com/watch?v=jILgxeNBK_8

Traveler’s dilemma

An airline manager asks two travelers, who lost their suitcases, to write down an amount between \$2 and \$100 inclusive. If both write down the same amount, the manager will reimburse both travelers that amount. However, if one writes down a smaller number, it will be taken as the true dollar value, and both travelers will receive that amount along with a bonus: \$2 extra to the traveler who wrote down the lower value and \$2 deduction from the person who wrote down the higher amount.

Traveler's dilemma

Kauchik Basu,
"The Traveler's Dilemma: Paradoxes of Rationality in Game Theory";
American Economic Review, Vol. 84, No. 2, pages 391-395; May 1994.

Traveler's dilemma

		Billy				
		100	99	98	…	2
Alan	100	(100,100)	(97,101)	(96,100)	…	(0,4)
	99	(101,97)	(99,99)	(96,100)	…	(0,4)
	98	(100,96)	(100,96)	(98,98)	…	(0,4)
	…	…	…	…	…	…
	2	(4,0)	(4,0)	(4,0)	...	(2,2)

Traveler’s dilemma

		Billy				
		100	99	98	…	2
Alan	100	(100,100)	(97,101)	(96,100)	…	(0,4)
	99	(101,97)	(99,99)	(96,100)	…	(0,4)
	98	(100,96)	(100,96)	(98,98)	…	(0,4)
	…	…	…	…	…	…
	2	(4,0)	(4,0)	(4,0)	...	(2,2)

Traveler’s dilemma

When the upper limit is 3, the Traveler’s dilemma is similar to Prisoner's dilemma

		Billy	
		3	2
Alan	3	(3,3)	(0,4)
	2	(4,0)	(2,2)

Traveler’s dilemma

		Peter	
		Not	Con
John	Not	(1,1)	(5,0)
	Con	(0,5)	(3,3)

Prisoner's dilemma

Money sharing game

1. Five players put certain amount of money from $0 to $1,000 to a pool.
2. The total amount of money in the pool will be multiplied by 3.
3. The money in the pool is then distributed evenly to the players.

Money sharing game

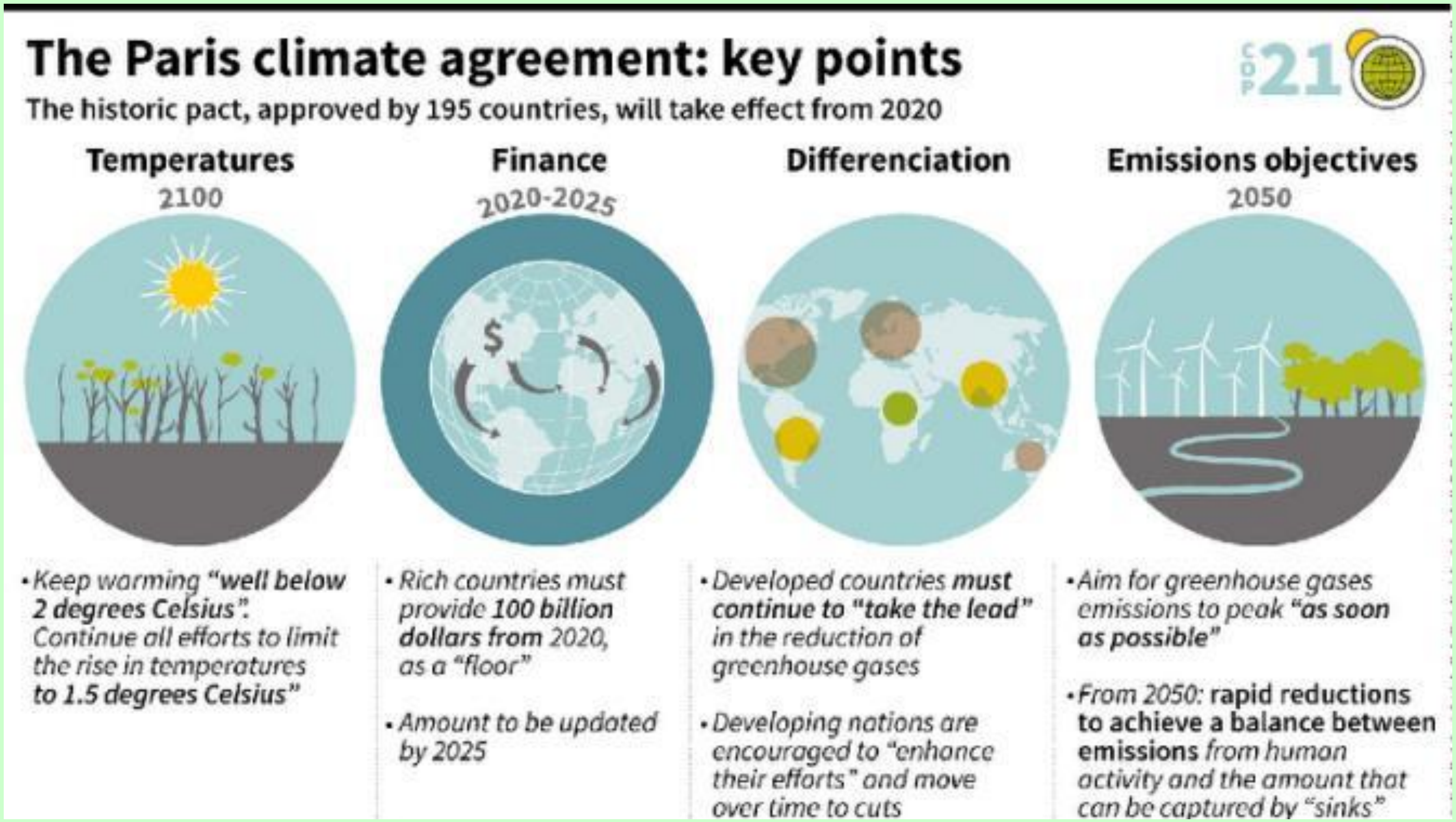
	Ideal Situation	Nash Equilibrium
Strategy	$1,000	$0
Payoff	$2,000	$0

No one will put money to the pool because every dollar a player puts become 3 dollars but will share evenly with 5 players.

Environment protection

The money sharing game explains why every country is blaming others instead of putting more resources to environmental protection.

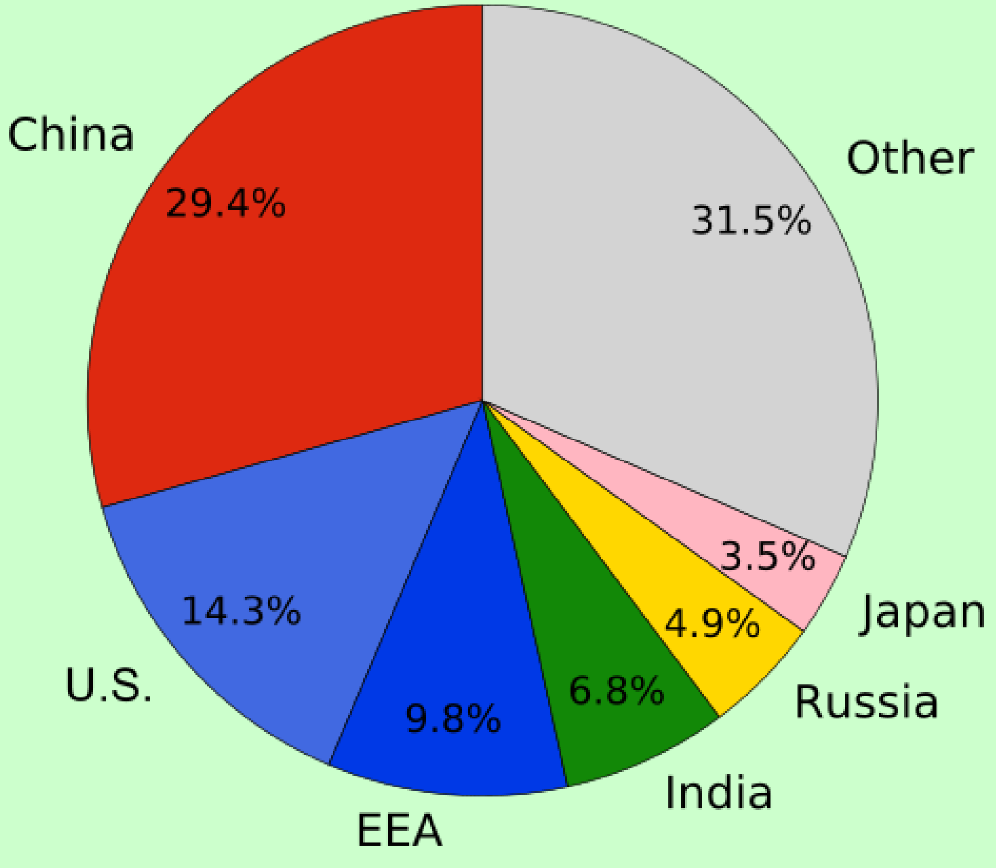
Paris climate agreement



US exit Paris agreement

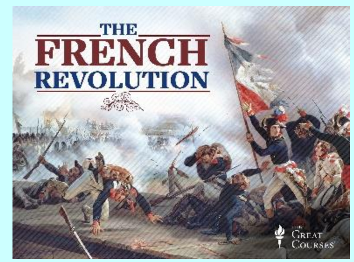
Trump (1 June 2017): The United State will withdraw from Paris climate accord.

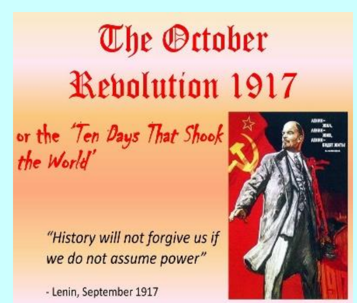
Global carbon dioxide emission

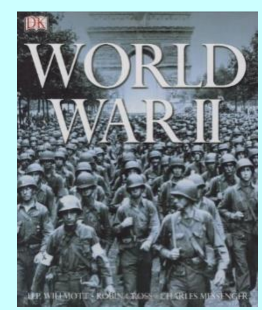


Game theory and politics

- Monarchy vs Republic
- Autocracy vs Democracy
- Union vs Independent
- Capitalism vs Socialism
- Liberalism vs Conservatism
- Egalitarianism vs Elitism
- Welfare vs Competition



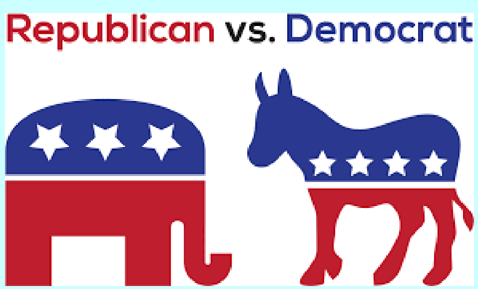




Game theory and politics

Players

- King vs Official vs People
- Government vs Rebellion
- Parties vs Parties
- Bourgeoisie vs Proletariat
- Country vs Country



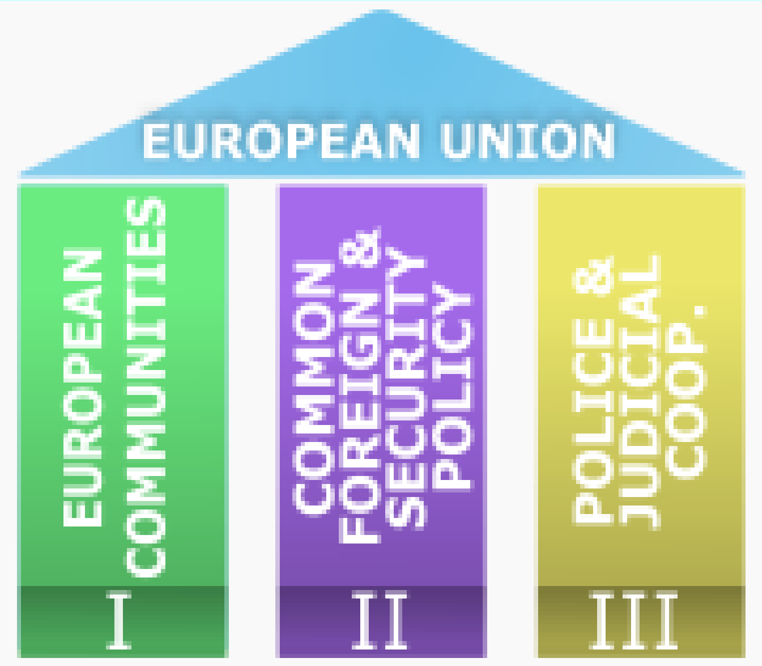
Brexit

History of European Union

1945: End of World War II
1951: European Coal and Steel Community was found in Treaty of Paris
1973: Accession of United Kingdom
1992: Born of European Union and European Monetary Union (UK not included) in Maastricht Treaty. Three pillars of EU established
2002: The euro replaces twelve national currencies
2009: Lisbon Treaty abolishes the three pillars of EU
2016: UK referendum to leave EU

Three Pillars of European Union

- Trade and other economic matters
- Common foreign and security policy
- Justice and home affairs

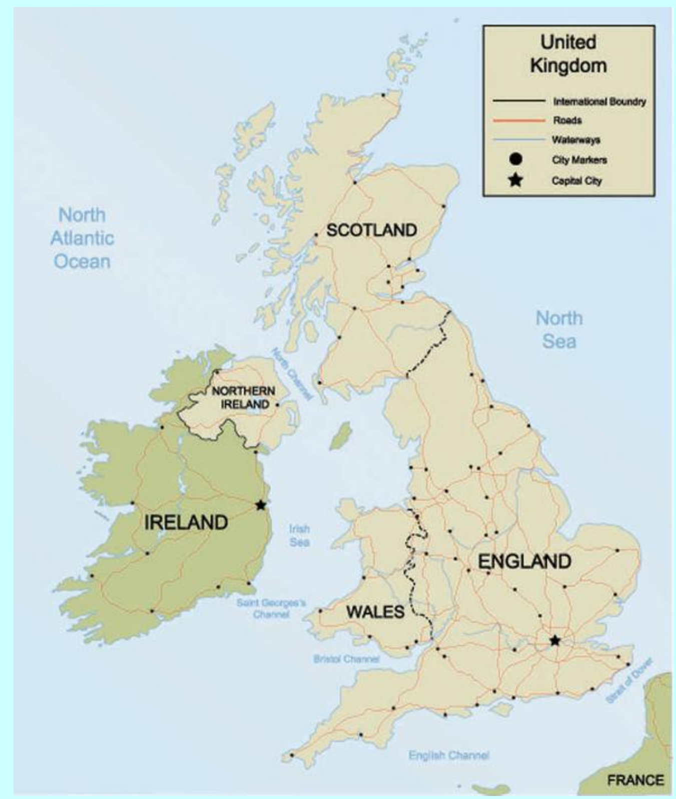


Brexit

	Pros	**Cons**
Trade	More power to negotiate trade agreement	Trade barriers with EU. Fall in export to EU country
Jobs	Less competition	Less jobs abroad
Membership fees	Saving billions of membership fee	Lost administrative support from EU
Foreign relation	Control of foreign policy	Less international influence
Immigrants	Control of immigrants	Lost migrant labor

Northern Ireland Backstop

- Make sure no hard border between Northern Ireland and Republic of Ireland if UK and EU can’t agree a trade deal after Brexit
- Limit UK doing trade deals around world.
- Rejected 3 times in UK parliament in early 2019



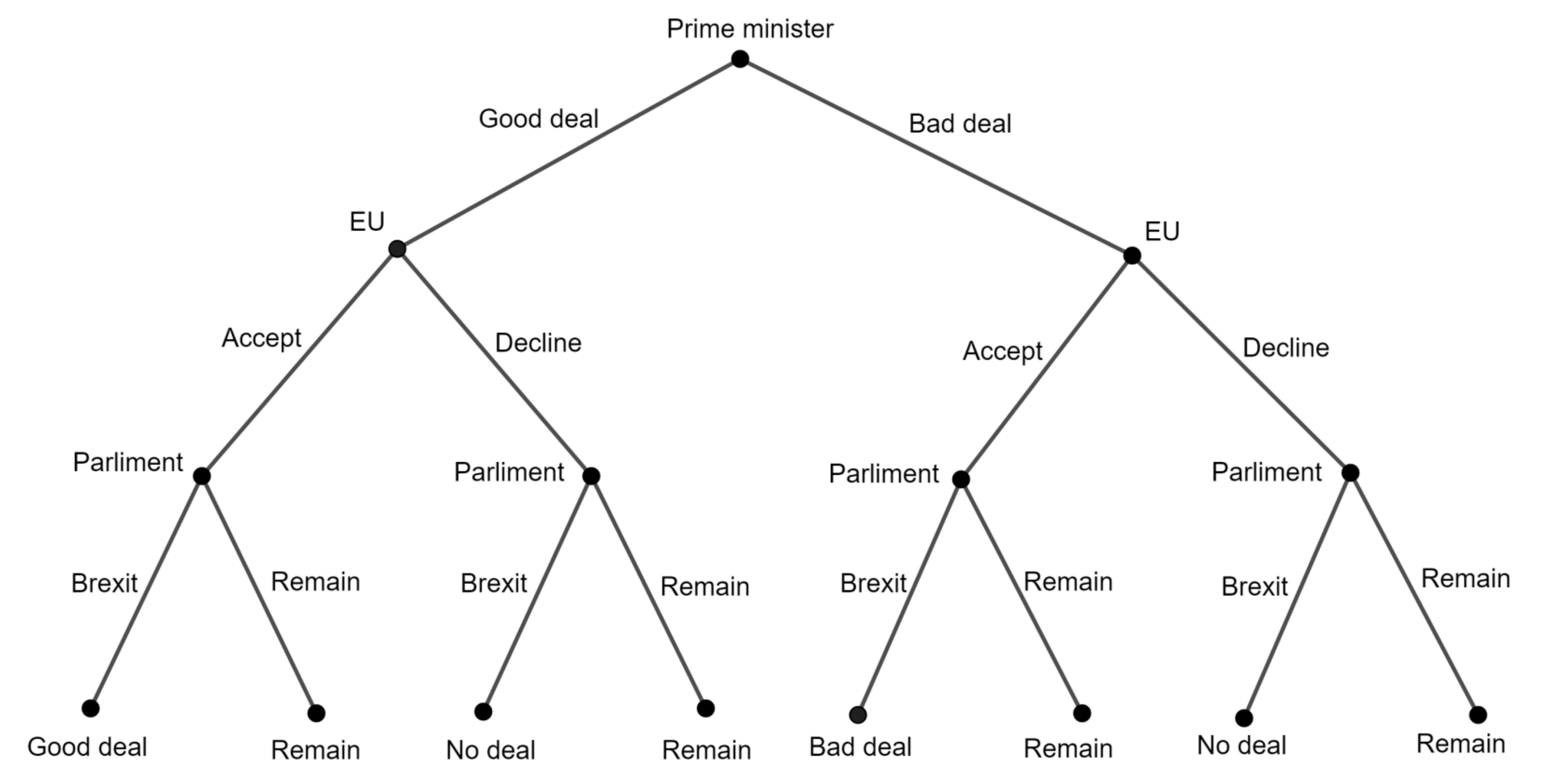
Boris Johnson's deal

- Mostly the same except backstop was dropped.
- Northern Ireland leaves EU customs union but remain partially aligned to single market.
- No customs checks on Island of Ireland
- Tariffs for goods from Great Britain to Northern Ireland will be possible

Preferences

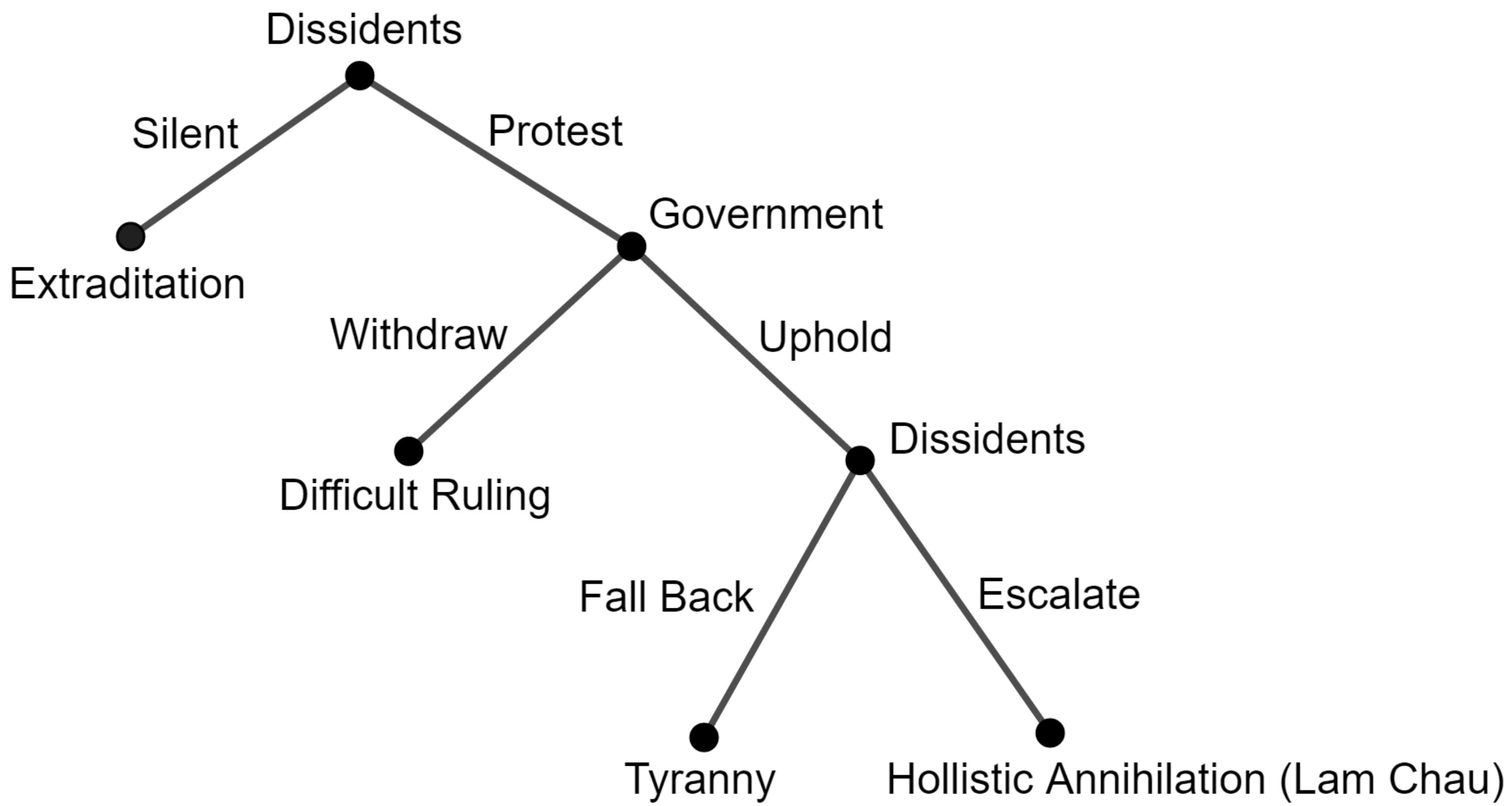
Theresa May	Boris Johnson	EU	Pro-brexit Party	Anti-brexit Party
Good deal	Good deal	Remain	Good deal	Remain
Bad deal	No deal	Bad deal	Bad deal	Good deal
Remain	Bad deal	Good deal	Remain	Bad deal
No deal	Remain	No deal	No deal	No deal

Brexit and Game Theory



Anti-Extradition Law Movement

Anti-Extradition Law Movement



Hong Kong Protests

2003: Anti-article 23

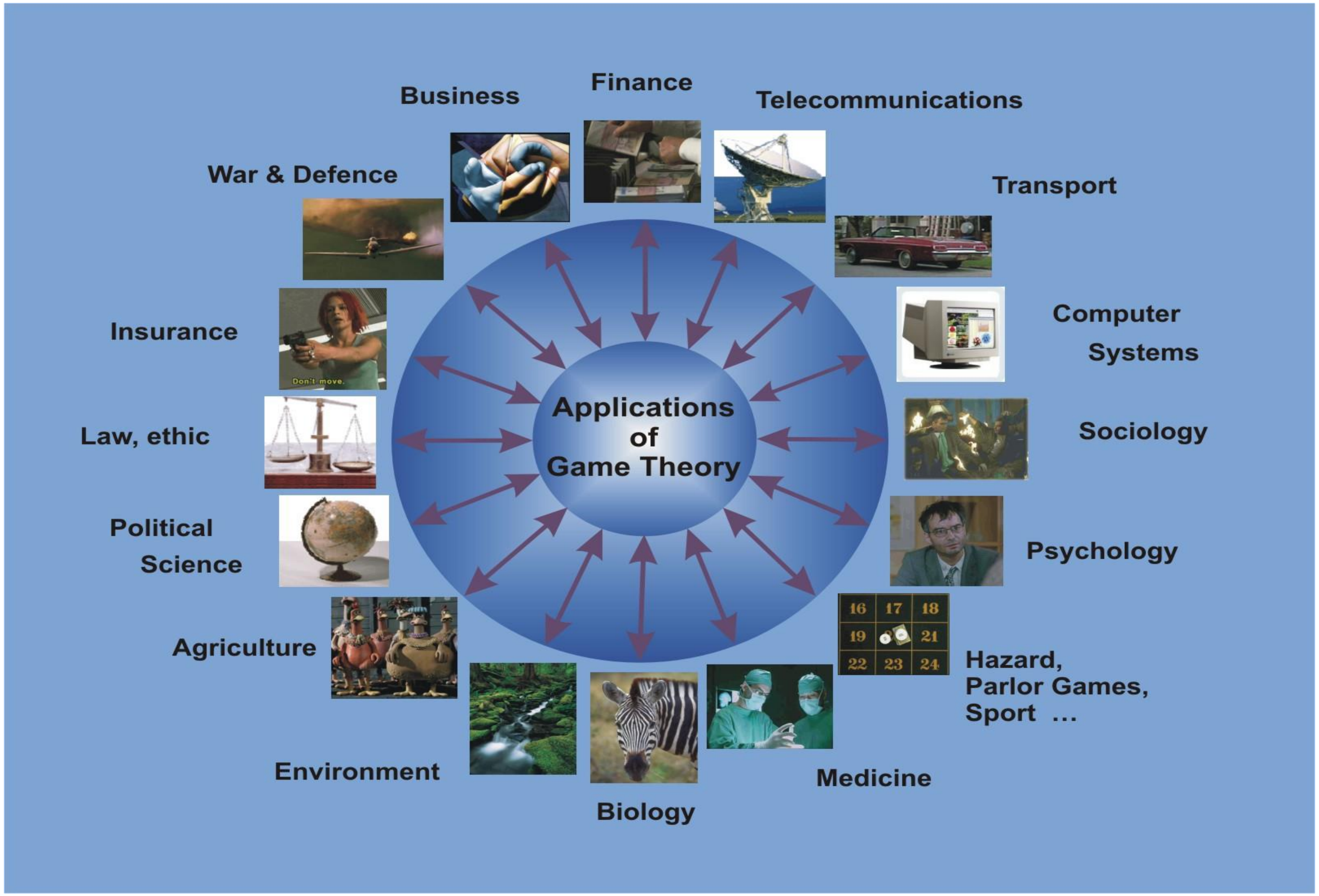
2010: Hong Kong electoral reform (for 2012)

2012: Anti-Moral and National Education movement

2013: Request issuing broadcasting licence for HKTV

2014: Umbrella movement

Application of Game Theory



Non-transferable utility

Cooperative game with non-transferable utility:

- A player cannot transfer its utility (payoff) to another player.
- The players may use joint strategy instead of using mixed strategy independently.

Joint strategy

Joint strategy:

Two players use varies pairs of strategies according to certain probabilities.

Examples:

1. Rock-scissors-paper:
   Using rock-rock with probability 0.7 and paper-scissors with probability 0.3.
2. Dating game:
   Watching soccer match with probability 0.1 and watching drama with probability 0.9.

Transferable utility

Cooperative game with transferable utility:

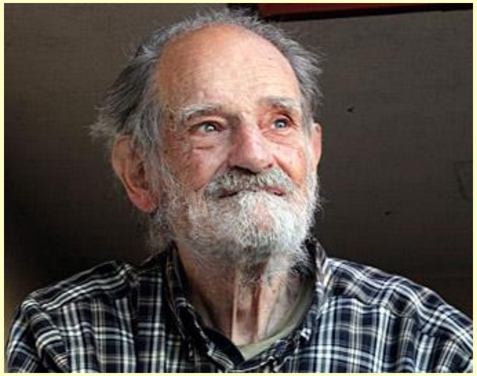
- A player can transfer its utility (payoff) to other players.
- The total payoff of the players is maximized.
- The players decide how to split the maximum total payoff.

Lloyd Stowell Shapley

- Born: 2 June 1923
  Dead: 12 March 2016
- His father Harlow Shapley is known for determining the position of the Sun in the Milky Way Galaxy

Lloyd Stowell Shapley

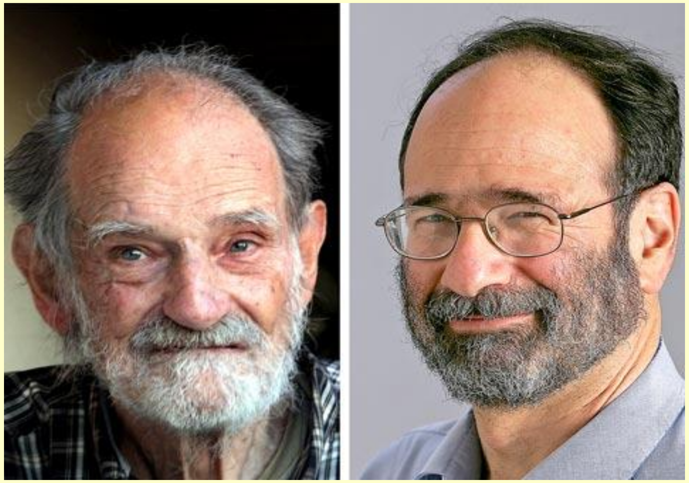
- Drafted when he was a student at Harvard in 1947



- Served in the Army in Chengdu, China and received the Bronze Star decoration for breaking the Soviet weather code

Nobel Prize in Economic 2012

- A value for *n*-person Games (1953)
- College Admissions and the Stability of Marriage (with Davis Gale 1962)
- Awarded Nobel Memorial Prize in Economic Sciences with Alvin Elliot Roth in 2012



Shapley Roth

Nobel Prize in Economic 2012

This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions.

Nobel Prize in Economic 2012

- I consider myself a mathematician and the award is for economics. I never, never in my life took a course in economics.

- The paper "College Admissions and the Stability of Marriage" was published after two initial rejections (for being too simple), and fifty years later in 2012 he won the Nobel Memorial Prize in Economic Sciences for the theory of stable allocation.

Stable marriage problem

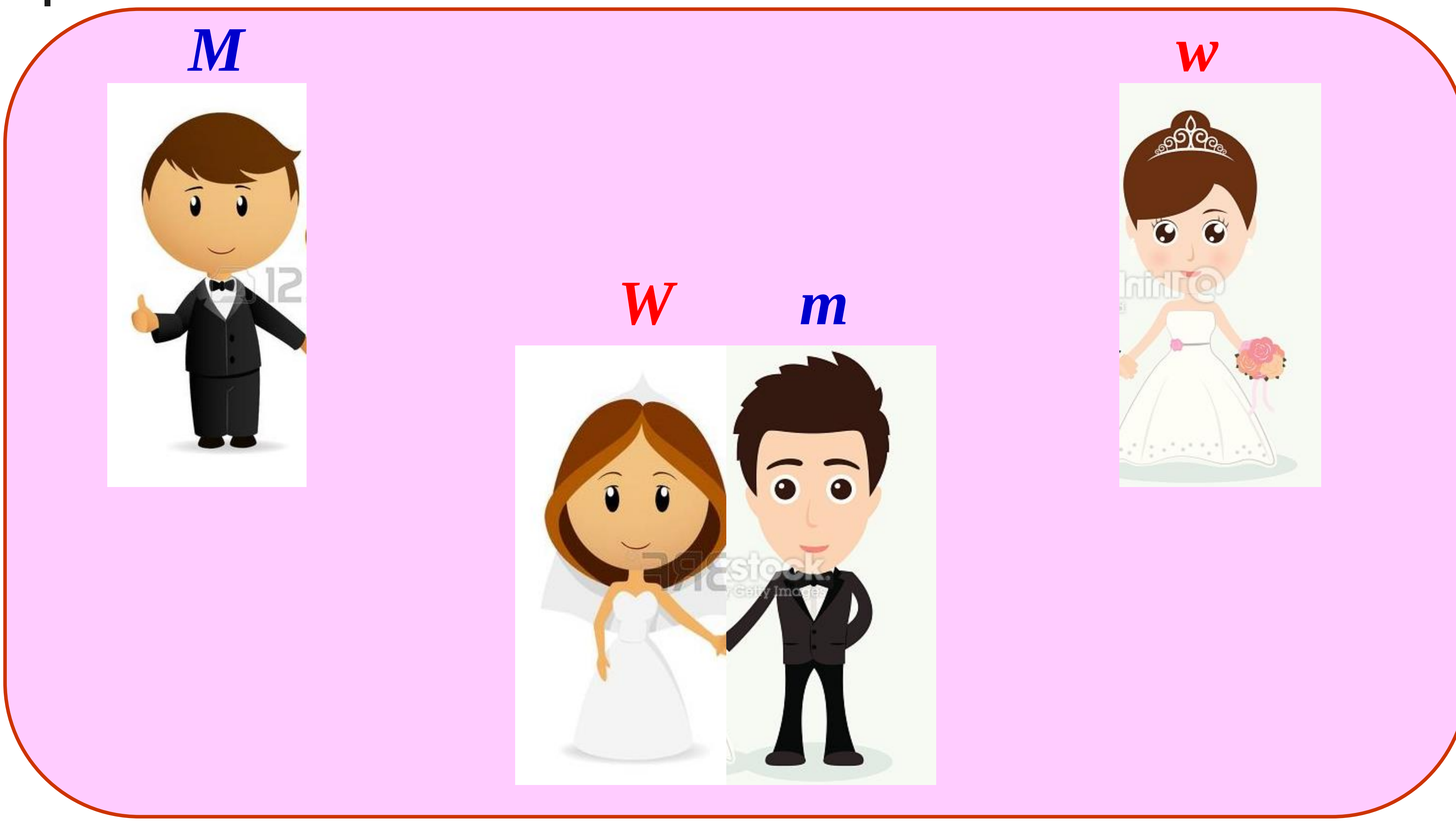
A set of marriages is unstable if there are two men *M* and *m* who are married to two women *W* and *w*, respectively, although *W* prefers *m* to *M* and *m* prefers *W* to *w*. A set of marriages is stable if it is not unstable.

Unstable set of marriages

M *W*

m *w*

Unstable set of marriages



Existence of stable marriage

Shapley's Theorem:

Suppose there are *n* men and *n* women. There always exists a stable set of marriages.

Ranking matrix

	W1	W2	W3
M1	**1,3**	**2,2**	**3,1**
M2	**3,1**	**1,3**	**2,2**
M3	**2,2**	**3,1**	**1,3**

- {(M1,W1), (M2,W2), (M3,W3)} is stable. (All men with their first choices.)
- {(M1,W3), (M2,W1), (M3,W2)} is stable. (All women with their first choices.)
- {(M1,W1), (M2,W3), (M3,W2)} is unstable. (Consider (M3,W1).)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	**1,2**	**2,1**	**3,2**	**4,1**
M2	**2,4**	**1,2**	**3,1**	**4,2**
M3	**2,1**	**3,3**	**4,3**	**1,4**
M4	**1,3**	**4,4**	**3,4**	**2,3**

Alternation of

- Men propose to their favorite women.
- Women reject unfavorable men.

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	**1,2**	**2,1**	**3,2**	**4,1**
M2	**2,4**	**1,2**	**3,1**	**4,2**
M3	**2,1**	**3,3**	**4,3**	**1,4**
M4	**1,3**	**4,4**	**3,4**	**2,3**

Step 1: Men propose to their favorite women.

(M1,W1),(M2,W2),(M3,W4),(M4,W1)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	1,2	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	1,4
M4	~~1,3~~	4,4	3,4	2,3

Step 2: Women reject unfavorable men.

(M1,W1),(M2,W2),(M3,W4),~~(M4,W1)~~

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	1,2	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	1,4
M4	~~1,3~~	4,4	3,4	2,3

Step 3: Men propose to their favorite women.

(M1,W1),(M2,W2),(M3,W4),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	1,2	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 4: Women reject unfavorable men.

(M1,W1),(M2,W2),~~(M3,W4)~~,(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	1,2	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 5: Men propose to their favorite women.

(M1,W1),(M2,W2),(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 6: Women reject unfavorable men.

~~(M1,W1)~~,(M2,W2),(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	2,4	1,2	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 7: Men propose to their favorable women.

(M1,W2),(M2,W2),(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	2,4	~~1,2~~	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 8: Women reject unfavorable men.

(M1,W2),~~(M2,W2)~~,(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	2,4	~~1,2~~	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 9: Men propose to their favorite women.

(M1,W2),(M2,W1),(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	~~2,4~~	~~1,2~~	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 10: Women reject unfavorable men.

(M1,W2),~~(M2,W2)~~,(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	~~2,4~~	~~1,2~~	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

Step 11: Men propose to their favorite women.

(M1,W2),(M2,W3),(M3,W1),(M4,W4)

Deferred-acceptance procedure

	W1	W2	W3	W4
M1	~~1,2~~	2,1	3,2	4,1
M2	~~2,4~~	~~1,2~~	3,1	4,2
M3	2,1	3,3	4,3	~~1,4~~
M4	~~1,3~~	4,4	3,4	2,3

A stable set of marriages is

(M1,W2),(M2,W3),(M3,W1),(M4,W4)

Note: This example has only one stable set.

Another example

	W1	W2	W3	W4
M1	3,1	1,3	4,1	2,4
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	1,4	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	1,3	4,1	2,4
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	1,4	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	~~1,3~~	4,1	2,4
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	~~1,4~~	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	~~1,3~~	4,1	2,4
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	~~1,4~~	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	~~1,3~~	4,1	~~2,4~~
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	~~1,4~~	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	~~1,3~~	4,1	~~2,4~~
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	~~1,4~~	4,2	2,3

Another example

	W1	W2	W3	W4
M1	3,1	~~1,3~~	4,1	~~2,4~~
M2	~~1,4~~	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	~~1,4~~	4,2	2,3

Another example

	W1	W2	W3	W4
M1	**3,1**	~~**1,3**~~	**4,1**	~~**2,4**~~
M2	~~**1,4**~~	**3,1**	**2,4**	**4,1**
M3	**4,2**	**1,2**	**2,3**	**3,2**
M4	**3,3**	~~**1,4**~~	**4,2**	**2,3**

A stable set of stable marriages is

(M1,W1),(M2,W3),(M3,W2),(M4,W4)

Another example

	W1	W2	W3	W4
M1	**3,1**	**1,3**	**4,1**	**2,4**
M2	**1,4**	**3,1**	**2,4**	**4,1**
M3	**4,2**	**1,2**	**2,3**	**3,2**
M4	**3,3**	**1,4**	**4,2**	**2,3**

Of course, we may ask the women to propose first.

Another example

	W1	W2	W3	W4
M1	3,1	1,3	~~4,1~~	2,4
M2	1,4	3,1	2,4	~~4,1~~
M3	4,2	1,2	2,3	3,2
M4	3,3	1,4	4,2	2,3

Then the men reject their unfavorable women.

Another example

	W1	W2	W3	W4
M1	**3,1**	**1,3**	~~**4,1**~~	**2,4**
M2	**1,4**	**3,1**	**2,4**	~~**4,1**~~
M3	**4,2**	**1,2**	**2,3**	**3,2**
M4	**3,3**	**1,4**	**4,2**	**2,3**

We obtain another stable set of marriages

(M1,W1),(M2,W2),(M3,W4),(M4,W3)

Another example

	W1	W2	W3	W4
M1	3,1	1,3	4,1	2,4
M2	1,4	3,1	2,4	4,1
M3	4,2	1,2	2,3	3,2
M4	3,3	1,4	4,2	2,3

We see that stable set of marriages is not unique

(M1,W1),(M2,W2),(M3,W4),(M4,W3)

(M1,W1),(M2,W3),(M3,W2),(M4,W4)

Problem of roommates

An even number of boys are divided up into pairs of roommates.

	B1	B2	B3	B4
B1		1,2	2,1	3,1
B2	2,1		1,2	3,2
B3	1,2	2,1		3,3
B4	1,3	2,3	3,3	

The boy pairs with B4 will have a better option.
Stable set of pairing does not always exist.

Shapley value

The Shapley value of player k is defined as

$$\phi_k = \sum_{S \subset N} \frac{(|S|-1)!(n-|S|)!}{n!} \delta(k,S)$$

where

$$\delta(k,S) = v(S) - v(S \setminus \{k\})$$

is the contribution of player k to coalition S.

Shapley's value of player k is the average contribution of player k to all orders of coalitions.