Solutions of MATH5360 Assignment 3

1. (a) Consider

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 5-4y \\ 3+y \end{pmatrix},$$

we have

$$\begin{cases} 5-4y > 3+y \ if \ 0 \leq y < 2/5, \\ 5-4y = 3+y \ if \ y = 2/5, \\ 5-4y < 3+y \ if \ 1 \geq y > 2/5. \end{cases}$$

Thus,

$P = \{(x,y) : (x = 0 \cap 1 \geq y > 2/5) \cup (0 \leq x \leq 1 \cap y = 2/5) \cup (x = 1 \cap 0 \leq y < 2/5)\}.$

Consider

$$\mathbf{x}B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2x+2 & -2x+3 \end{pmatrix},$$

we have

$$\begin{cases} 2x+2 < -2x+3 \ if \ 0 \leq x < 1/4, \\ 2x+2 = -2x+3 \ if \ x = 1/4, \\ 2x+2 > -2x+3 \ if \ 1 \geq x > 1/4. \end{cases}$$

Thus,

$Q = \{(x,y) : (0 \leq x < 1/4 \cap y = 0) \cup (x = 1/4 \cap 0 \leq y \leq 1) \cup (1/4 < x \leq 1 \cap y = 1)\}.$

By Figure 1, $P \cap Q = \{(1/4, 2/5)\}$. Therefore, the game has a Nash equilibrium $(\mathbf{p}, \mathbf{q}) = \big((1/4, 3/4), (2/5, 3/5)\big)$ and the payoff for row player is $5 - 4y = 17/5$, the payoff for column player is $2x + 2 = 5/2$.

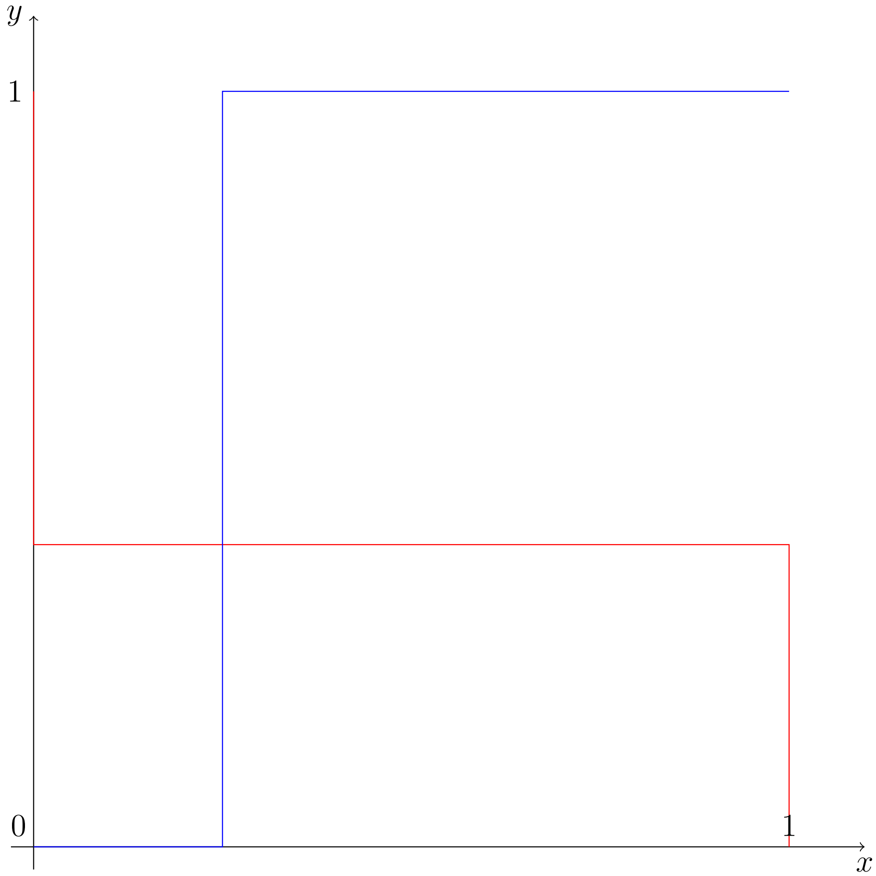

Figure 1:

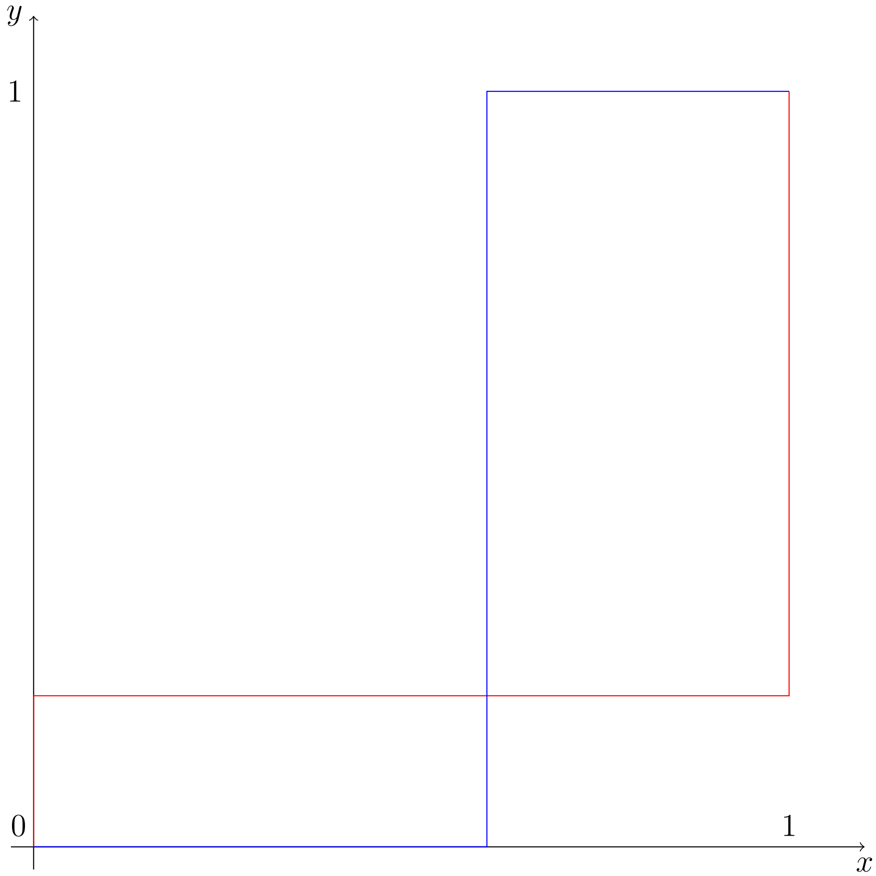

Figure 2:

(b)Since

$$P = \{(x, y) : (x = 0 \cap 0 \leq y < 1/5) \cup (0 \leq x \leq 1 \cap y = 1/5) \cup (x = 1 \cap 1/5 < y \leq 1)\},$$

and

$$Q = \{(x, y) : (0 \leq x < 3/5 \cap y = 0) \cup (x = 3/5 \cap 0 \leq y \leq 1) \cup (3/5 < x \leq 1 \cap y = 1)\},$$

by Figure 2, $P \cap Q = \{(0, 0), (3/5, 1/5), (1, 1)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0, 1), (0, 1)\big), \big((3/5, 2/5), (1/5, 4/5)\big), \big((1, 0), (1, 0)\big)\}$ and the payoff pair is $\{(3, 4), (13/5, 8/5), (5, 2)\}$.

(c)Since

$$P = \{(0, y) : 0 \leq y \leq 1)\},$$

and

$$Q = \{(x, y) : (0 < x \leq 1 \cap y = 1) \cup (x = 0 \cap 0 \leq y \leq 1)\},$$

$P \cap Q = \{(0, y) : 0 \leq y \leq 1)\}$. Therefore, the game has infinite Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0, 1), (y, 1-y) : y \in [0, 1]\big)\}$ and the payoff pair is $\{(4 + y, 2) : y \in [0, 1]\}$.

(d)Since

$$P = \{(x, y) : (x = 0 \cap 1/2 < y \leq 1) \cup (0 \leq x \leq 1 \cap y = 1/2) \cup (x = 1 \cap 0 \leq y < 1/2)\},$$

and

$$Q = \{(x, y) : (0 \leq x < 4/5 \cap y = 1) \cup (x = 4/5 \cap 0 \leq y \leq 1) \cup (4/5 < x \leq 1 \cap y = 0)\},$$

by Figure 3, $P \cap Q = \{(0, 1), (4/5, 1/2), (1, 0)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0, 1), (1, 0)\big), \big((4/5, 1/5), (1/2, 1/2)\big), \big((1, 0), (0, 1)\big)\}$ and the payoff pair is $\{(4, 3), (1/2, 3/5), (2, 1)\}$.

2.(a)

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 5 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix}.$$

Since the 1st column of B is dominated by the 2nd column of B, $\mathbf{q} = (0, y, 1 - y)$. Consider

$$\begin{pmatrix} 4 & 2 & 3 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3-y \\ 1+4y \end{pmatrix},$$

we have

$$\begin{cases} 3 - y > 1 + 4y \ if \ 0 \leq y < 2/5, \\ 3 - y = 1 + 4y \ if \ y = 2/5, \\ 3 - y < 1 + 4y \ if \ 1 \geq y > 2/5. \end{cases}$$

Thus,

$$P = \{(x, y) : (x = 1 \cap 0 \leq y < 2/5) \cup (0 \leq x \leq 1 \cap y = 2/5) \cup (x = 0 \cap 1 \geq y > 2/5)\}.$$

Consider

$$\mathbf{x}B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 2-x & -2x+5 & 1+3x \end{pmatrix},$$

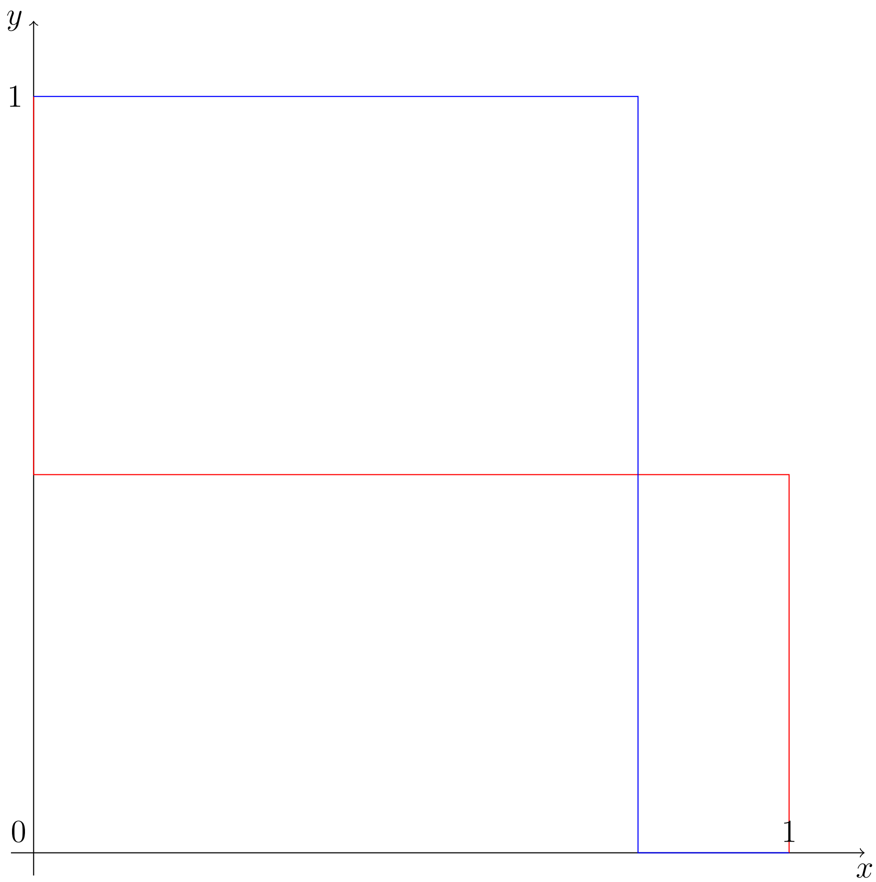

Figure 3:

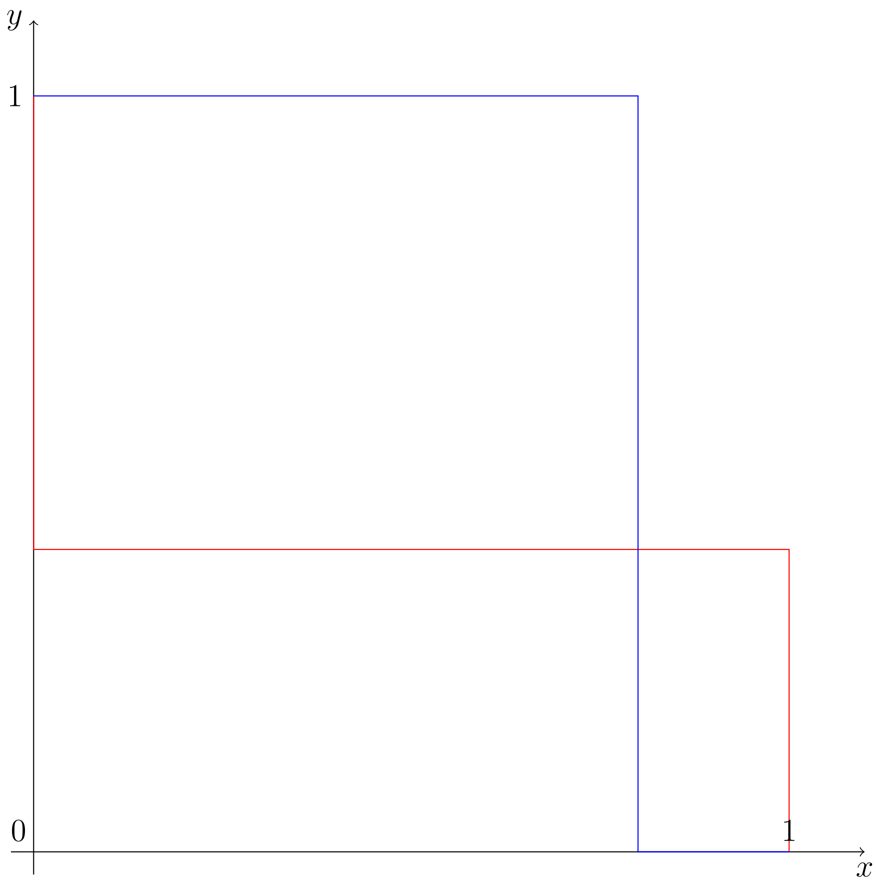

Figure 4:

we have

$$\begin{cases} 5-2x > 1+3x \ if \ 0 \leq x < 4/5, \\ 5-2x = 1+3x \ if \ x = 4/5, \\ 5-2x < 1+3x \ if \ 1 \geq x > 4/5. \end{cases}$$

Thus,

$Q = \{(x,y) : (0 \leq x < 4/5 \cap y = 1) \cup (x = 4/5 \cap 0 \leq y \leq 1) \cup (1 \geq x > 4/5 \cap y = 0)\}.$

by Figure 4, $P \cap Q = \{(0,1), (4/5, 2/5), (1,0)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0,1),(0,1,0)\big), \big((4/5,1/5),(0,2/5,3/5)\big), \big((1,0),(0,0,1)\big)\}$ and the payoff pair is $\{(5,5), (13/5, 13/5), (3,4)\}$.

(b)

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 1 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Since the 2nd column of B is dominated by the 1st column of B, $\mathbf{q} = (y, 0, 1-y)$. Consider

$$\begin{pmatrix} 1 & 4 & 5 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} y \\ 0 \\ 1-y \end{pmatrix} = \begin{pmatrix} 5-4y \\ 2+y \end{pmatrix},$$

we have

$$\begin{cases} 5-4y > 2+y \; if \; 0 \le y < 3/5, \\ 5-4y > 2+y \; if \; y = 3/5, \\ 5-4y < 2+y \; if \; 1 \ge y > 3/5. \end{cases}$$

Thus,

$$P = \{(x,y) : (x = 1 \cap 0 \le y < 3/5) \cup (0 \le x \le 1 \cap y = 3/5) \cup (x = 0 \cap 1 \ge y > 3/5)\}.$$

Consider

$$\mathbf{x}B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2-2x & 1-2x & 2x-1 \end{pmatrix},$$

we have

$$\begin{cases} 2-2x > 2x-1 \; if \; 0 \le x < 3/4, \\ 2-2x = 2x-1 \; if \; x = 3/4, \\ 2-2x < 2x-1 \; if \; 1 \ge x > 3/4. \end{cases}$$

Thus,

$$Q = \{(x,y) : (0 \le x < 3/4 \cap y = 1) \cup (x = 3/4 \cap 0 \le y \le 1) \cup (1 \ge x > 3/4 \cap y = 0)\}.$$

by Figure 5, $P \cap Q = \{(0,1), (3/4, 3/5), (1,0)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0,1),(1,0,0)\big), \big((3/4,1/4),(3/5,0,2/5)\big), \big((1,0),(0,0,1)\big)\}$ and the payoff pair is $\{(3,2),(2.6,0.5),(5,1)\}$.

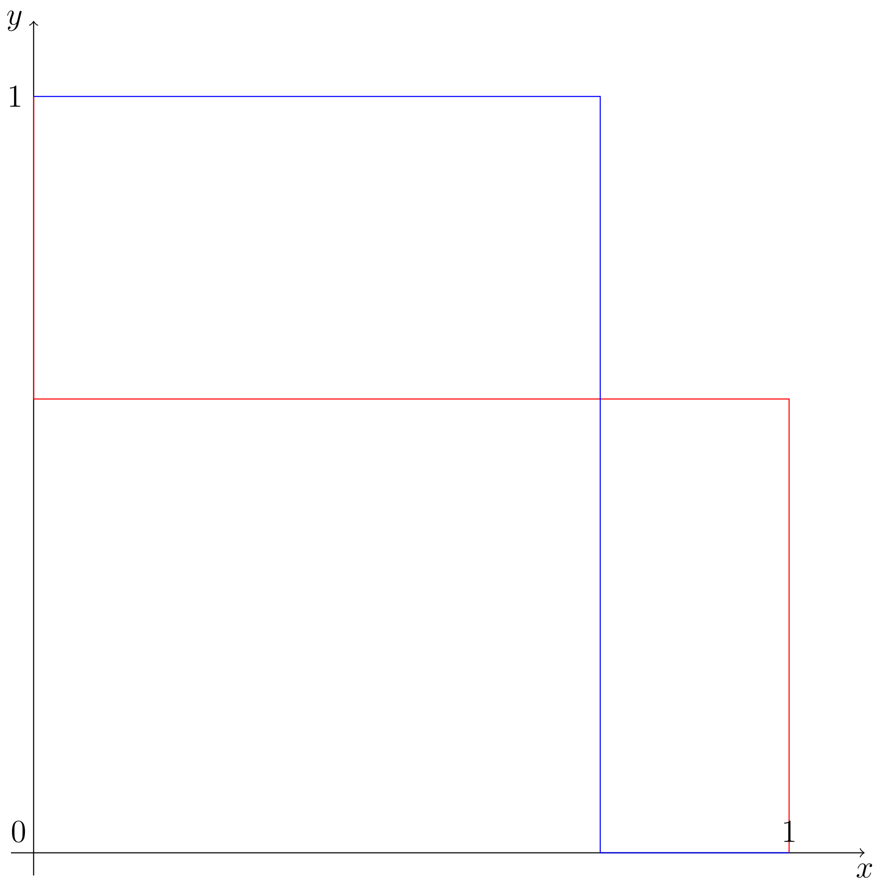

Figure 5:

(c)

$$A = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 6 & -1 \\ 5 & 1 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 6 & 3 & -1 \\ 4 & 5 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Since the 1st row of A is dominated by the 3rd row of A, $\mathbf{p} = (0, x, 1-x)$. A and B can be reduced to

$$A' = \begin{pmatrix} 2 & 6 & -1 \\ 5 & 1 & 4 \end{pmatrix},$$

$$B' = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Since the 1st column of B' is dominated by the 2nd column of B', $\mathbf{q} = (0, y, 1-y)$. A' and B' can be reduced to

$$A'' = \begin{pmatrix} 6 & -1 \\ 1 & 4 \end{pmatrix},$$

$$B'' = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}.$$

Consider

$$\begin{pmatrix} 6 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 7y-1 \\ 4-3y \end{pmatrix},$$

we have

$$\begin{cases} 7y-1 > 4-3y \; if \; 1 \geq y > 1/2, \\ 7y-1 = 4-3y \; if \; y = 1/2, \\ 7y-1 < 4-3y \; if \; 0 \leq y < 1/2. \end{cases}$$

Thus,

$$P = \{(x,y) : (x = 1 \cap 1 \geq y > 1/2) \cup (0 \leq x \leq 1 \cap y = 1/2) \cup (x = 0 \cap 0 \leq y < 1/2)\}.$$

Consider

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2+3x & 3-2x \end{pmatrix},$$

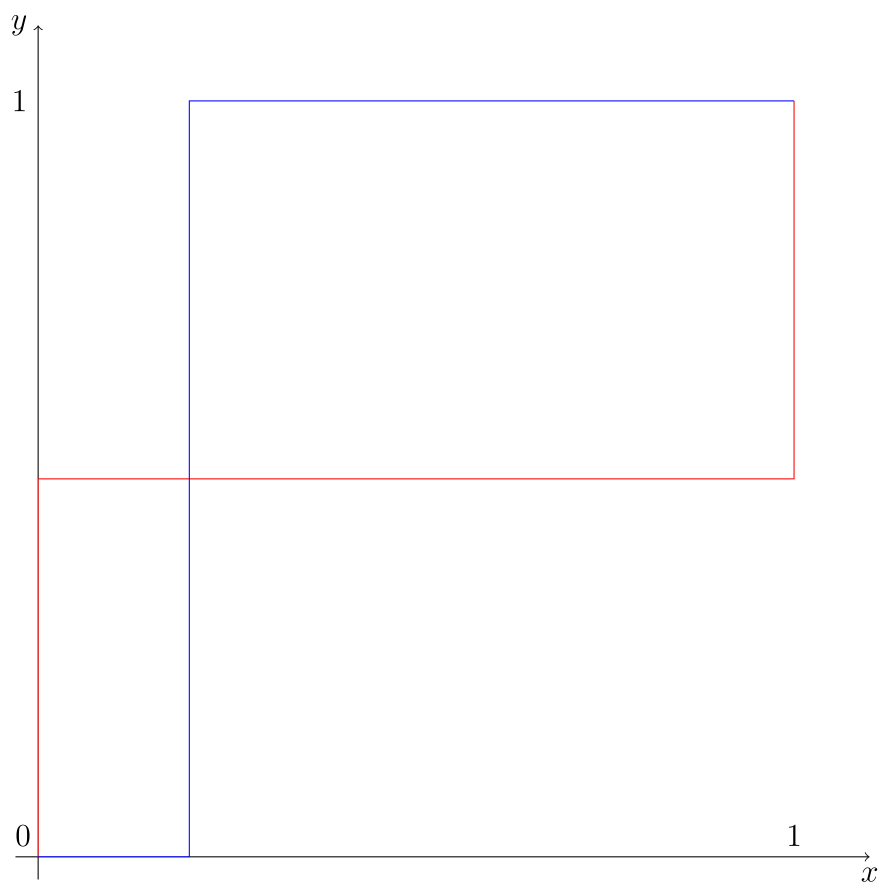

Figure 6:

we have

$$\begin{cases} 2+3x > 3-2x \ if \ 1 \geq x > 1/5, \\ 2+3x = 3-2x \ if \ x = 1/5, \\ 2+3x < 3-2x \ if \ 0 \leq x < 1/5. \end{cases}$$

Thus,

$Q = \{(x, y) : (1 \geq x > 1/5 \cap y = 1) \cup (x = 1/5 \cap 0 \leq y \leq 1) \cup (0 \leq x < 1/5 \cap y = 0)\}.$

by Figure 6, $P \cap Q = \{(0, 0), (1/5, 1/2), (1, 1)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0, 0, 1), (0, 0, 1)\big), \big((0, 1/5, 4/5), (0, 1/2, 1/2)\big), \big((0, 1, 0), (0, 1, 0)\big)\}$ and the payoff pair is $\{(4, 3), (2.5, 2.6), (6, 5)\}$.

(d)

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 8 & 3 \\ 5 & 5 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 0 & 9 \\ 6 & 4 & 5 \\ 4 & 3 & 1 \end{pmatrix}.$$

Since the 2nd column of B is dominated by the 1st column of B, $\mathbf{q} = (y, 0, 1-y)$. A and B can be reduced to

$$A' = \begin{pmatrix} 3 & 7 \\ 2 & 3 \\ 5 & 4 \end{pmatrix},$$

$$B' = \begin{pmatrix} 2 & 9 \\ 6 & 5 \\ 4 & 1 \end{pmatrix}.$$

Since the 2nd row of A' is dominated by the 1st row of A', $\mathbf{p} = (x, 0, 1-x)$. A' and B' can be reduced to

$$A'' = \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix},$$

$$B'' = \begin{pmatrix} 2 & 9 \\ 4 & 1 \end{pmatrix}.$$

Consider

$$\begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 7-4y \\ 4+y \end{pmatrix},$$

we have

$$\begin{cases} 7-4y > 4+y \; if \; 0 \leq y < 3/5, \\ 7-4y = 4+y \; if \; y = 3/5, \\ 7-4y < 4+y \; if \; 1 \geq y > 3/5. \end{cases}$$

Thus,

$$P = \{(x,y) : (x = 1 \cap 0 \leq y < 3/5) \cup (0 \leq x \leq 1 \cap y = 3/5) \cup (x = 0 \cap 1 \geq y > 3/5)\}.$$

Consider

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 2 & 9 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 4-2x & 1+8x \end{pmatrix},$$

we have

$$\begin{cases} 4-2x > 1+8x \ if \ 0 \le x < 0.3, \\ 4-2x = 1+8x \ if \ x = 0.3, \\ 4-2x < 1+8x \ if \ 1 \ge x > 0.3. \end{cases}$$

Thus,

$$Q = \{(x,y) : (0 \le x < 0.3 \cap y = 1) \cup (x = 0.3 \cap 0 \le y \le 1) \cup (1 \ge x > 0.3 \cap y = 0)\}.$$

by Figure 7, $P \cap Q = \{(0,1), (0.3, 0.6), (1,0)\}$. Therefore, the game has three Nash equilibriums $(\mathbf{p}, \mathbf{q}) = \{\big((0,0,1),(1,0,0)\big), \big((0.3,0,0.7),(0.6,0,0.4)\big), \big((1,0,0),(0,0,1)\big)\}$ and the payoff pair is $\{(5,4), (4.6, 3.4), (7,9)\}$.

3.(a) Define $f : (x,y) \mapsto (-y, x)$. If f has a fixed point (x_0, y_0), then $-y_0 = x_0$ and $x_0 = y_0$ which implies $(x_0, y_0) = (0,0) \notin X$. Hence f has no fixed point in X.

(b) Define $f : (x,y,z) \mapsto (y, -z, x)$. If f has a fixed point (x_0, y_0, z_0), then $y_0 = x_0$, $-z_0 = y_0$ and $x_0 = z_0$ which implies $(x_0, y_0, z_0) = (0,0,0) \notin X$. Hence f has no fixed point in X.

(c) Define $f : (x,y) \mapsto (\frac{x+1}{2}, \frac{y}{2})$. If f has a fixed point (x_0, y_0), then $\frac{x_0+1}{2} = x_0$ and $\frac{y_0}{2} = y_0$ which implies $(x_0, y_0) = (1,0) \notin X$. Hence f has no fixed point in X.

4.(a)

$$A = \begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix},$$

$$B^T = \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$(\mu, \nu) = (\nu_A, \nu B^T) = (2/3, -1).$$

See Figure 8. The equation of the line segment joining $(0,1)$ and $(1,0)$ is given by $v = 1-u$ and $g(u,v) = (u-2/3)(v+1) = -u^2 + \frac{8}{3}u - \frac{4}{3}$, $u \in [2/3, 1]$, which attains its maximum at $u = 1$. The equation of the line segment joining $(4,-4)$ and $(1,0)$ is given by $v = -\frac{4}{3}(u-1)$ and $g(u,v) = (u-2/3)(v+1) = -\frac{4}{3}u^2 + \frac{29}{9}u - \frac{14}{9}$, $u \in [1,4]$, which attains its maximum at $u = \frac{29}{24}$. Thus, the arbitration pair is $(\alpha, \beta) = (\frac{29}{24}, -\frac{5}{18})$.

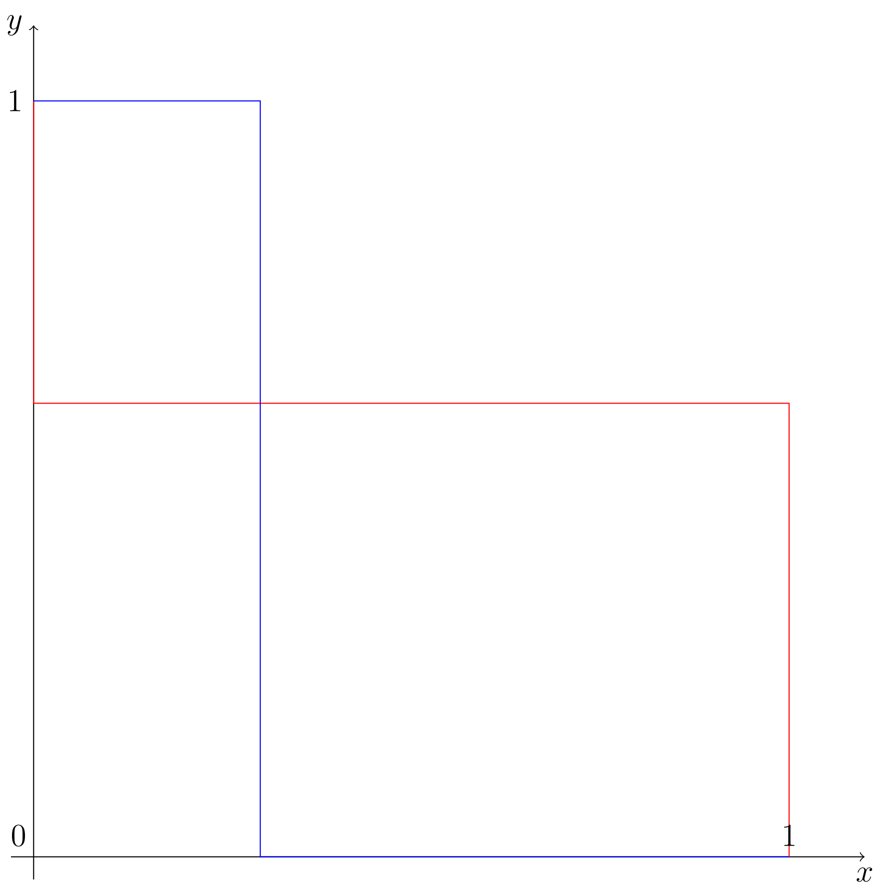

Figure 7:

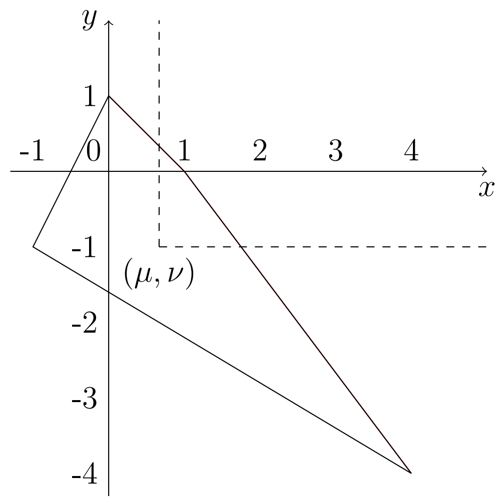

Figure 8:

(b)

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix},$$

$$B^T = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

$$(\mu, \nu) = (\nu_A, \nu B^T) = (3/2, 3/5).$$

See Figure 9. The equation of the line segment joining $(2, 3)$ and $(3, 1)$ is given by $v = -2(u-3)+1$ and $g(u, v) = (u-1.5)(v-0.6) = (u-1.5)(-2u+6.4)$, $u \in [2, 3]$, which attains its maximum at $u = 2.35$. Thus, the arbitration pair is $(\alpha, \beta) = (2.35, 2.3)$.

(c)

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix},$$

$$B^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 3 \end{pmatrix}.$$

$$(\mu, \nu) = (\nu_A, \nu B^T) = (0, 7/5).$$

See Figure 10. The equation of the line segment joining $(1, 3)$ and $(4, 1)$ is given by $v = -\frac{2}{3}(u-4) + 1$ and $g(u, v) = (u-0)(v-1.4) = u(-\frac{2}{3}u + \frac{34}{15})$,

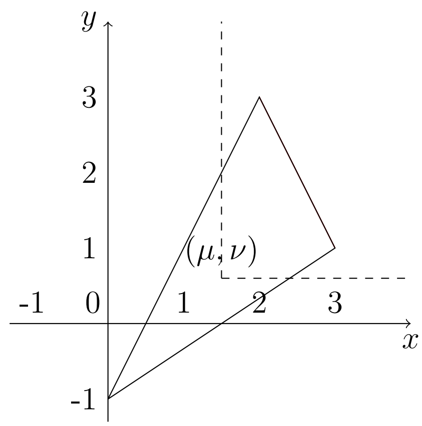

Figure 9:

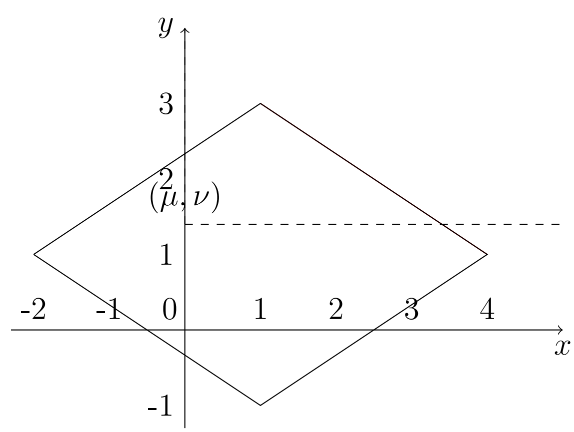

Figure 10:

$u \in [1,4]$, which attains its maximum at $u = 1.7$. Thus, the arbitration pair is $(\alpha, \beta) = (1.7, \frac{38}{15})$.

(c)

$$A = \begin{pmatrix} 6 & 0 & 4 \\ 8 & 4 & 0 \end{pmatrix},$$

$$B^T = \begin{pmatrix} 4 & -2 \\ 10 & 1 \\ 1 & 3 \end{pmatrix}.$$

$$(\mu, \nu) = (\nu_A, \nu B^T) = (2, 1).$$

See Figure 11. The equation of the line segment joining $(6,4)$ and $(8,-2)$ is given by $v = -3u + 22$ and $g(u,v) = (u-2)(v-1) = -3u^2 + 27u - 42$, $u \in [6,8]$, which attains its maximum at $u = 6$. The equation of the line segment joining $(6,4)$ and $(0,10)$ is given by $v = -u + 10$ and $g(u,v) = (u-2)(v-1) = (u-2)(-u+9)$, $u \in [0,6]$, which attains its maximum at $u = 5.5$. Thus, the arbitration pair is $(\alpha, \beta) = (5.5, 4.5)$.

5. If we regard NTV and CTV as the row player and the column player respectively, then the bimatrix game is

$$A = \begin{pmatrix} 20 & 50 \\ 0 & 0 \end{pmatrix},$$

$$B^T = \begin{pmatrix} 0 & 40 \\ 0 & 0 \end{pmatrix}.$$

$$(\mu, \nu) = (\nu_A, \nu B^T) = (20, 0).$$

See Figure 12. The equation of the line segment joining $(50,0)$ and $(0,40)$ is given by $v = -\frac{4}{5}u + 40$ and $g(u,v) = (u-20)(v-0) = -\frac{4}{5}u^2 + 56u - 800$, $u \in [0,50]$, which attains its maximum at $u = 35$. Thus, the arbitration pair is $(\alpha, \beta) = (35, 12)$.

6. (a) The bargaining set is shown in Figure 13. On the bargaining set,

$$g(u,v) = (u-0)(v-0) = uv = u(4-u^2).$$

Since $g'(u) = 4 - 3u^2, g'(u) = 0 \Longrightarrow u = \frac{2\sqrt{3}}{3}$. It is easy to see g attains its maximum at $u = \frac{2\sqrt{3}}{3}$ on the bargaining set. Hence in this case, we have arbitration pair $(\frac{2\sqrt{3}}{3}, \frac{8}{3})$.

(b) When $(\mu, \nu) = (0, 1)$, the bargaining is shown in Figure 14. In this

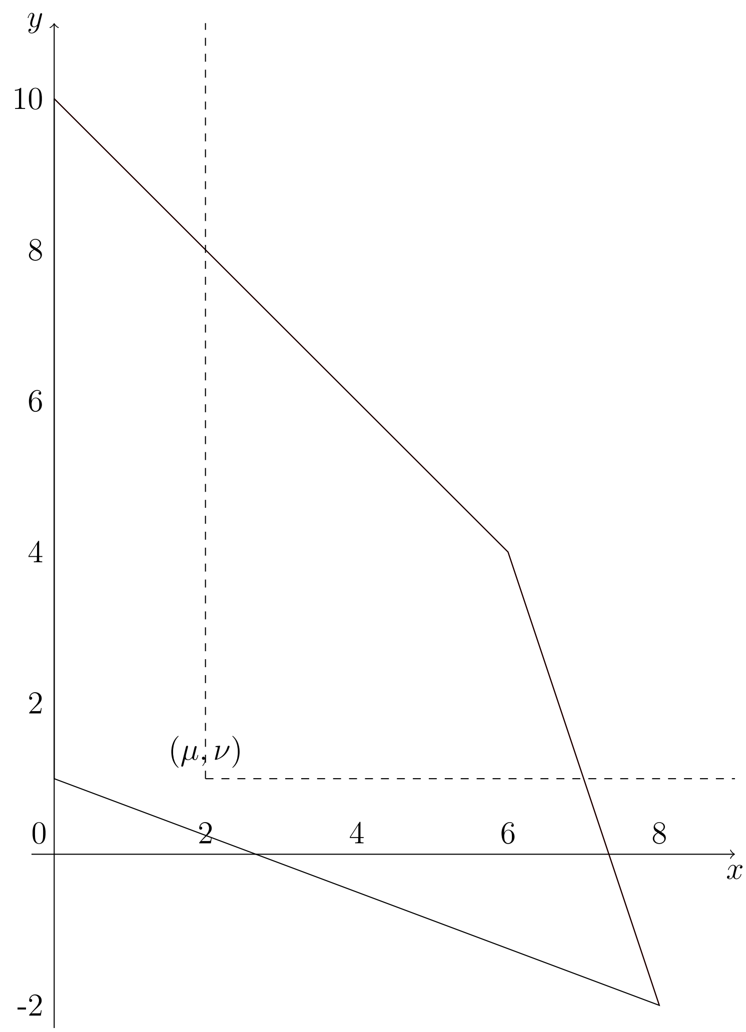

Figure 11:

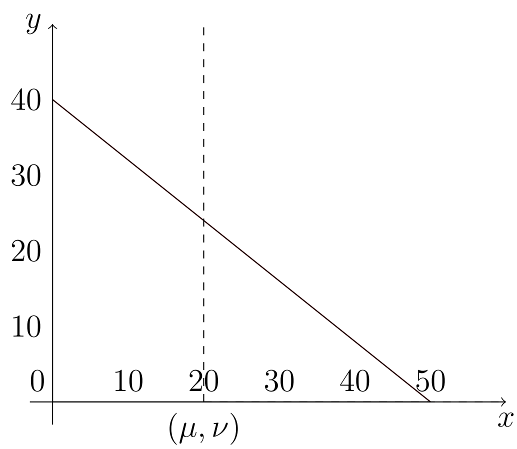

Figure 12:

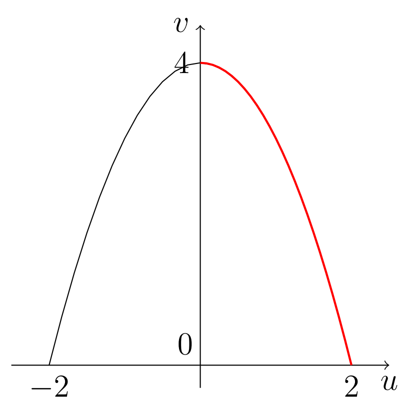

Figure 13:

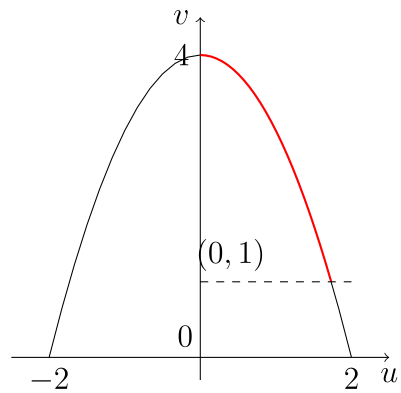

Figure 14:

case, on the bargaining set,

$$g(u,v) = (u-0)(v-1) = u(4-u^2-1) = 3u-u^3.$$

Let $g'(u) = 0$, we get $u = 1$. The arbitration pair is $(1,3)$.

7. Let $g(u,v) = (u-\mu)(v-\nu)$ on $\mathcal{R}$. In the intersection of the bargaining set and a neighborbood of (α,β), we have

$$g(u,v) = (u-\mu)(f(u)-\nu) := h(u).$$

Since g attains its maximum at (α,β), we have $h'(\alpha) = 0$, which implies easily that $f'(\alpha) = -\frac{\beta-\nu}{\alpha-\mu}$.

8. (a) Set $\mathbf{p} = \mathbf{q} = (1/n, 1/n, ..., 1/n)$, then $A\mathbf{q}^T = (r, r, ..., r)^T$ and $\mathbf{p}A^T = (r, r, ..., r)$ and, by the definition of Nash equilibrium, $(\mathbf{p},\mathbf{q})$ is an equilibrium pair with (r,r) as payoff pair.

(b) Let m be the maximum entry of $\dfrac{A+A^T}{2}$ and $a_{ij} \in A$ such that $\dfrac{a_{ij}+a_{ji}}{2} = m$. Denote $\nu(A)$ by a, then $(\mu,\nu) = (a,a)$. By the definition of m, the line segment joining (a_{ij}, a_{ji}) and (a_{ji}, a_{ij}) is belong to the bargaining set of (A, A^T) and is given by $u+v = 2m$. On this line segment, $g(u,v) = (u-a)(v-a) = (u-a)(2m-u-a)$ attains its maximum at $u = m$. On other part of the bargaining set, $u+v \le 2m$ and $g(u,v) \le (m-a)^2$. Therefore, the arbitration payoff pair of the bimatrix (A, A^T) is $(\alpha,\beta) = (m,m)$.

9.(a) The maximum total payoff is $2+4 = 6$ and the threat matrix is

$$T = A - B = \begin{pmatrix} 5 & -2 \\ 1 & 4 \end{pmatrix}.$$

The threat strategies are $\mathbf{p}_d = (\frac{4-1}{5-(-2)-1+4}, \frac{5-(-2)}{5-(-2)-1+4}) = (0.3, 0.7)$ and $\mathbf{q}_d = (\frac{4-(-2)}{5-(-2)-1+4}, \frac{5-1}{5-(-2)-1+4}) = (0.6, 0.4)$. The threat differential is $\delta = \frac{5\times4-(-2)\times1}{5-(-2)-1+4} = 2.2$ and the threat solution is $(\varphi_1, \varphi_2) = (\frac{6+2.2}{2}, \frac{6-2.2}{2}) = (4.1, 1.9)$.

(b) The maximum total payoff is $5 + 3 = 8$ and the threat matrix is

$$T = A - B = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}.$$

The threat strategies are $\mathbf{p}_d = (\frac{1-0}{2-(-2)-0+1}, \frac{2-(-2)}{2-(-2)-0+1}) = (0.2, 0.8)$ and $\mathbf{q}_d = (\frac{1-(-2)}{2-(-2)-0+1}, \frac{2-0}{2-(-2)-0+1}) = (0.6, 0.4)$. The threat differential is $\delta = \frac{2\times1-(-2)\times0}{2-(-2)-0+1} = 0.4$ and the threat solution is $(\varphi_1, \varphi_2) = (\frac{8+0.4}{2}, \frac{8-0.4}{2}) = (4.2, 3.8)$.

(c) The maximum total payoff is $4 + 7 = 11$ and the threat matrix is

$$T = A - B = \begin{pmatrix} 2 & -1 & -3 \\ -4 & 2 & 1 \end{pmatrix}.$$

The second column is strictly dominated by the last. The threat strategies are then easily determined to be $\mathbf{p}_d = (0.5, 0.5)$ and $\mathbf{q}_d = (0.4, 0, 0.6)$. The threat differential is $\delta = -1$ and the threat solution is $(\varphi_1, \varphi_2) = (\frac{11+(-1)}{2}, \frac{11-(-1)}{2}) = (5, 6)$.

(d) The maximum total payoff is $7 + 5 = 12$ and the threat matrix is

$$T = A - B = \begin{pmatrix} -6 & 2 & 3 \\ -7 & 1 & 0 \\ 4 & -8 & -5 \end{pmatrix}.$$

The second row is strictly dominated by the first row and T can be reduced to

$$T' = \begin{pmatrix} -6 & 2 & 3 \\ 4 & -8 & -5 \end{pmatrix}.$$

The last column is strictly dominated by the second column. The threat strategies are then easily determined to be $\mathbf{p}_d = (0.6, 0, 0.4)$ and $\mathbf{q}_d = (0.5, 0.5, 0)$. The threat differential is $\delta = -2$ and the threat solution is $(\varphi_1, \varphi_2) = (\frac{12+(-2)}{2}, \frac{12-(-2)}{2}) = (5, 7)$.