Solutions of MATH5360 Assignment 1

1. (a)

$$\begin{array}{lcccc} min\backslash max & 5 & 2 & 5 & 6 \\ -2 & \multicolumn{4}{c}{} \\ -2 & & & & \\ 2 & & & & \end{array} \quad \begin{array}{l} \\ \begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix} \end{array}$$

Both the maximin and minimax are 2. Therefore the entry $a_{32} = 2$ is a saddle point and the value of the game is 2.

(b)

$$\begin{array}{lcccc} min\backslash max & 2 & 5 & 3 & -1 \\ -4 & & & & \\ -1 & & & & \\ -5 & & & & \\ -4 & & & & \end{array} \quad \begin{array}{l} \\ \begin{pmatrix} -4 & 5 & -3 & -3 \\ 0 & 1 & 3 & -1 \\ -3 & -1 & 2 & -5 \\ 2 & -4 & 0 & -2 \end{pmatrix} \end{array}$$

Both the maximin and minimax are -1. Therefore the entry $a_{24} = -1$ is a saddle point and the value of the game is -1.

2. (a) By the Theorem 2.2.2, the maximin strategy for the row player is (2/5,3/5), the minimax strategy for the column player is (9/10,1/10) and the value of the game is 8/5.

(b) By the Theorem 2.2.2, the maximin strategy for the row player is (3/5,2/5), the minimax strategy for the column player is (1/2,1/2) and the value of the game is 1.

(c)Draw the graph of

$$\begin{cases} C_1 : v = 3x - 2(1-x) = 5x - 2 \\ C_2 : v = 2x + (1-x) = 1 + x \\ C_3 : v = 4x - 4(1-x) = 8x - 4 \\ C_4 : v = 5(1-x) = 5 - 5x \end{cases}.$$

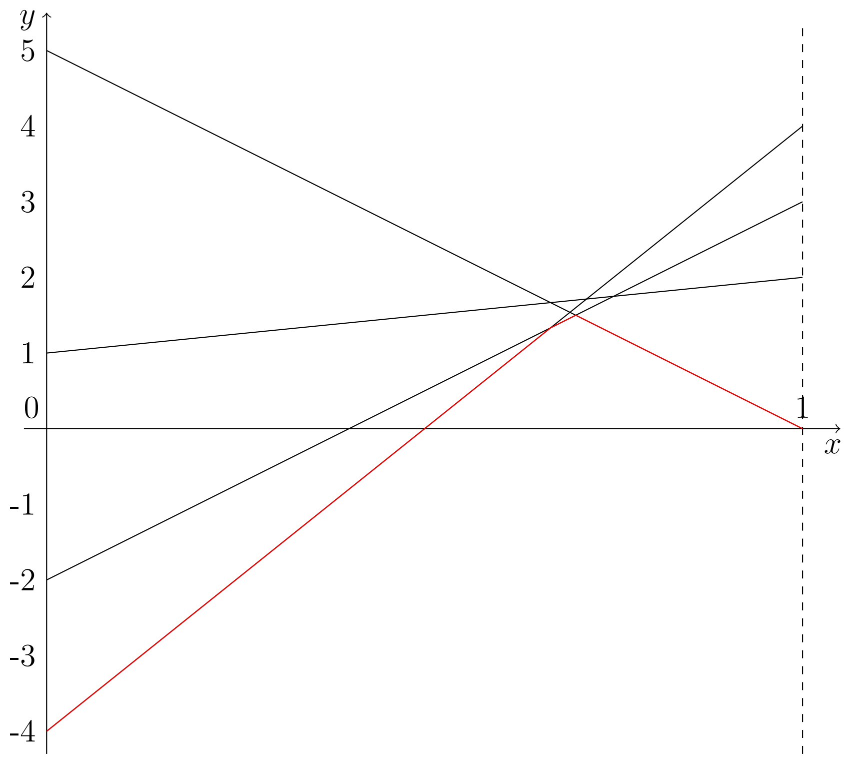

Figure 1:

The lower envelope is shown in Figure 1. Solving

$$\begin{cases} C1 : v = 5x - 2 \\ C4 : v = 5 - 5x \end{cases},$$

we have $v = 3/2$ and $x = 0.7$. Hence $v(A) = 3/2$ and the optimal strategy for the row player is $(0.7, 0.3)$. Solving

$$\begin{pmatrix} 3 & 0 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix},$$

we have $y_1 = y_4 = 0.5$. Therefore, the maximin strategy for the row player is (0.7,0.3), the minimax strategy for the column player is (0.5,0,0,0.5) and the value of the game is 3/2.
(d)Draw the graph of

$$\begin{cases} C_1 : v = x \\ C_2 : v = 2(1 - x) = 2 - 2x \\ C_3 : v = 4x - 3(1 - x) = 7x - 3 \\ C_4 : v = 2x - 2(1 - x) = 4x - 2 \end{cases}.$$

The lower envelope is shown in Figure 2. Solving

$$\begin{cases} C1 : v = x \\ C2 : v = 2 - 2x \\ C4 : v = 4x - 2 \end{cases},$$

we have $v = 2/3$ and $x = 2/3$. Hence $v(A) = 2/3$ and the optimal strategy for the row player is $(2/3, 1/3)$. Solving

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix},$$

we have $(y_1, y_2, y_4) = (2/3 - 2t, 1/3 + t, t)$ with $t \in [0, 1/3]$. Therefore, the maximin strategy for the row player is $(2/3, 1/3)$, the minimax strategy for the column player is $(2/3 - 2t, 1/3 + t, 0, t)$ with $t \in [0, 1/3]$ and the value of the game is $2/3$.

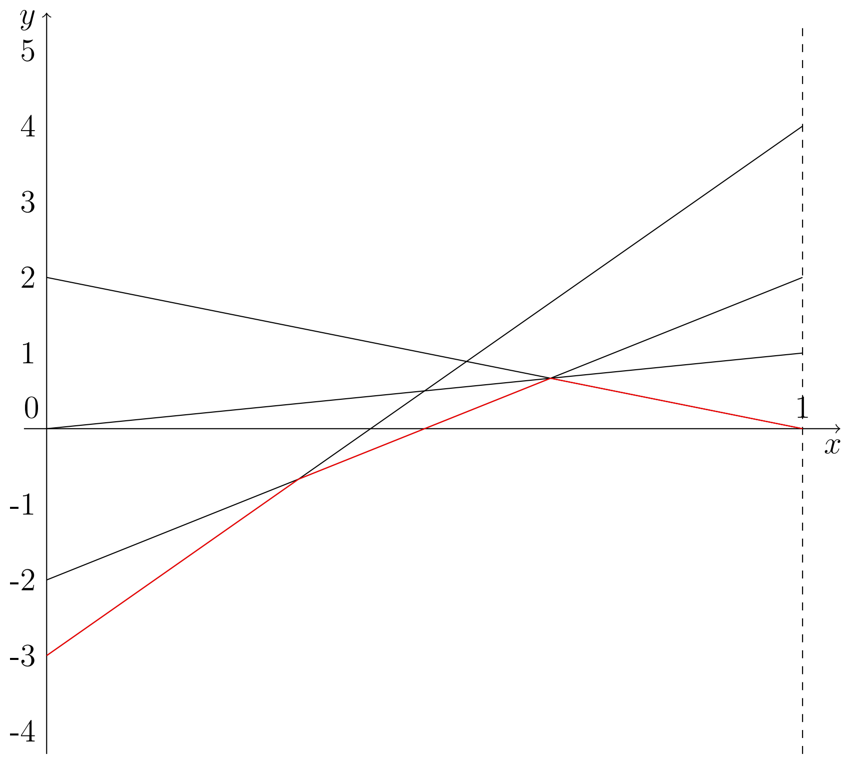

Figure 2:

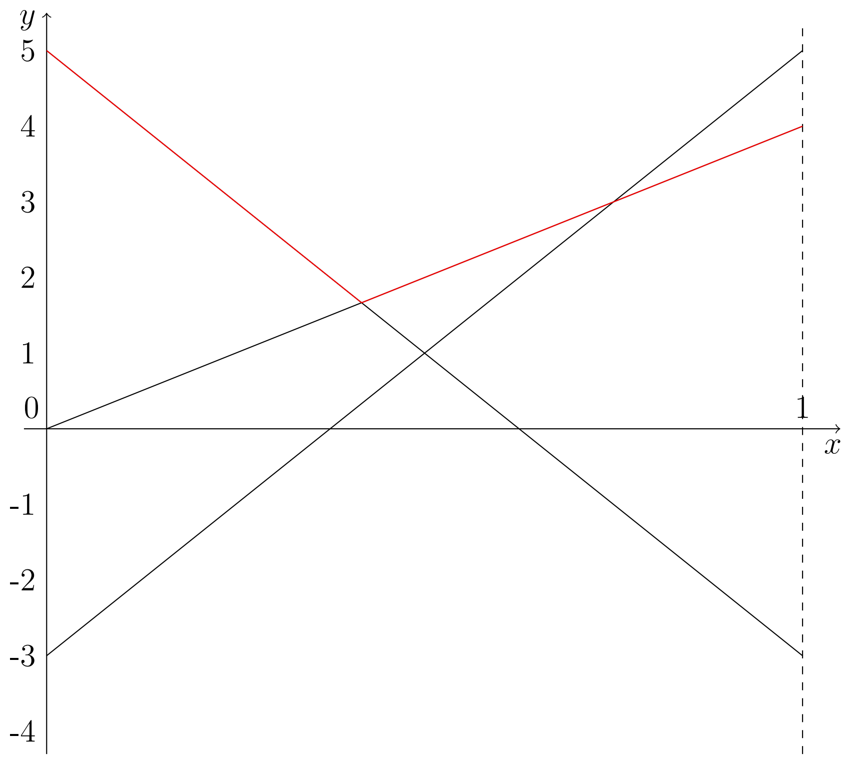

Figure 3:

(e)The third row is dominated by the fourth row, by deleting the third row we obtain

$$\begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 4 & 0 \end{pmatrix}.$$

Draw the graph of

$$\begin{cases} C_1 : v = 5x - 3(1-x) = 8x - 3 \\ C_2 : v = -3x + 5(1-x) = 5 - 8x \\ C_3 : v = 4x \end{cases}.$$

The upper envelope is shown in Figure 3. Solving

$$\begin{cases} C2 : v = 5 - 8x \\ C3 : v = 4x \end{cases},$$

we have $v = 5/3$ and $x = 5/12$. Hence $v(A) = 5/3$ and the optimal strategy for the column player is $(5/12, 7/12)$. Solving

$$\begin{pmatrix} x_2 & x_4 \end{pmatrix} \begin{pmatrix} -3 & 5 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 5/3 & 5/3 \end{pmatrix},$$

we have $(x_2, x_4) = (1/3, 2/3)$. Therefore, the maximin strategy for the row player is $(0, 1/3, 0, 2/3)$, the minimax strategy for the column player is $(5/12, 7/12)$ and the value of the game is $5/3$.

(f)The second row is dominated by the first row, by deleting the second row we obtain

$$\begin{pmatrix} 5 & -2 & 4 \\ 0 & 3 & 2 \end{pmatrix}.$$

Draw the graph of

$$\begin{cases} C_1 : v = 5x \\ C_2 : v = -2x + 3(1-x) = 3 - 5x \\ C_3 : v = 4x + 2(1-x) = 2 + 2x \end{cases}.$$

The lower envelope is shown in Figure 4. Solving

$$\begin{cases} C1 : v = 5x \\ C2 : v = 3 - 5x \end{cases},$$

we have $v = 1.5$ and $x = 0.3$. Hence $v(A) = 1.5$ and the optimal strategy for the row player is $(0.3, 0, 0.7)$. Solving

$$\begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix},$$

we have $(y_1, y_2) = (0.3, 0.7)$. Therefore, the maximin strategy for the row player is $(0.3, 0, 0.7)$, the minimax strategy for the column player is $(0.3, 0.7, 0)$ and the value of the game is 1.5.

(g)

$$\begin{array}{lc} min\backslash max & \begin{matrix} 5 & 2 & 5 & 6 \end{matrix} \\ -2 & \\ -2 & \begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix} \\ 2 & \end{array}$$

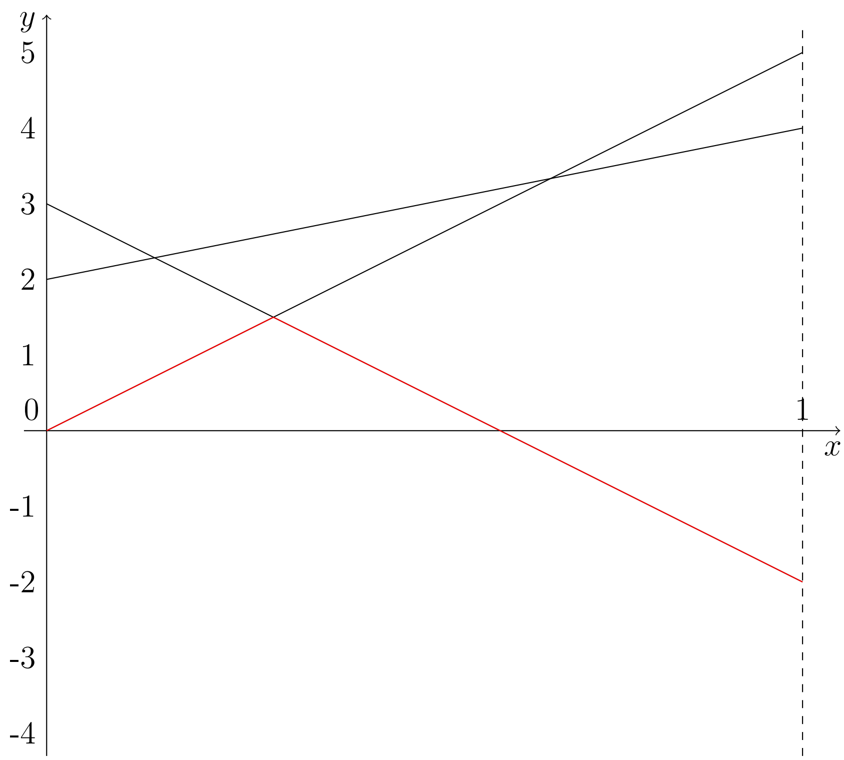

Figure 4:

Both the maximin and minimax are 2. Therefore the entry $a_{32} = 2$ is a saddle point and the value of the game is 2. The maximin strategy for the row player is $(0, 0, 1)$, the minimax strategy for the column player is $(0, 1, 0, 0)$

3. The game matrix is

$$A = \begin{pmatrix} -3 & 2 \\ 9 & -8 \end{pmatrix}.$$

By the Theorem 2.2.2, the maximin strategy for Raymond is (17/22,5/22), the minimax strategy for Calvin is (5/11,6/11) and the value of the game is -3/11.

4. By regarding Alex as the row player and Becky as the column player, the game matrix is

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 11 \end{pmatrix}.$$

(a) By the Theorem 2.2.2, the maximin strategy for Alex is (3/4,1/4), the minimax strategy for Becky is (3/4,1/4) and the value of the game is 2.

(b) $k = v = 2$.

5. The game matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The first row and the last row are dominated by the third row and fifth row respectively, by deleting the first row and the last row we obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The third, fourth and fifth columns are dominated by the first, second and last columns respectively, by deleting the third, fourth and fifth columns we obtain

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The third row is dominated by the first row, by deleting the third row we obtain

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the principle of indifference, the value of A_3 is $1/4$ and the optimal strategies for player I and II are $(1/4, 1/4, 1/4, 1/4)$. Therefor, the value of A is $1/4$ and the optimal strategies for player I is $(0, 1/4, 1/4, 0, 1/4, 1/4, 0)$, the optimal strategies for player II is $(1/4, 1/4, 0, 0, 0, 1/4, 1/4)$.

6.

$$A^{-1} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solve $v\vec{1} \cdot A^{-1} \cdot \vec{1} = 1$ we obtain $v = 1/12$. The optimal strategies are $p = \vec{v} \cdot A^{-1} = (1/12, 1/4, 1/3, 1/3)$ and $q^T = \vec{v}(A^{-1})^T = (1/3, 1/3, 1/4, 1/12)^T$.

7. The game matrix is

$$A = \begin{pmatrix} 0 & -1 & 2 & 2 & 2 \\ 1 & 0 & -1 & 2 & 2 \\ -2 & 1 & 0 & -1 & 2 \\ -2 & -2 & 1 & 0 & -1 \\ -2 & -2 & -2 & 1 & 0 \end{pmatrix}.$$

The matrix is skewed symmetric, so the value is 0. The fourth row(column) and the last row(column) are dominated by the first row(column), by deleting the fourth row(column) and the last row(column) we obtain

$$A_1 = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

By the principle of indifference, solve $\vec{p_1}A_1 = 0$ with $\vec{p_1}\cdot\vec{1} = 1$ and $A_1\vec{q_1}^T = 0$ with $\vec{q_1}\cdot\vec{1} = 1$, we obtain $p_1 = q_1 = (1/4, 1/2, 1/4)$. Therefore, the optimal strategies are $p = q = (1/4, 1/2, 1/4, 0, 0)$.

8. (a) The payoff matrix of Aaron is

$$A = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}.$$

(b) By the Theorem 2.2.2, the maximin strategy for Aaron is (2/5,3/5), the minimax strategy for Becky is (1/2,1/2) and the value of the game is 1.

9. (a) (1)If $c \le -3$, then the entry $a_{11} = -3$ is a saddle point.
(2)If $-3 < c \le -2$, then the entry $a_{21} = c$ is a saddle point.
(3)If $c > -2$, then A has no saddle point.
(b)The value of A is 0 implies A has no saddle point and, by the Theorem 2.2.2, $v = (c-6)/(6+c) = 0$, thus $c = 6$. And the maximin strategy for the row player is (2/3,1/3), the minimax strategy for the column player is (1/4,3/4).

10. Suppose $(v, \vec{p}, \vec{q})$ solves A, that is

$$\begin{cases} -\vec{p}A^T\vec{y}^T = \vec{p}A\vec{y}^T \ge v, \\ -\vec{x}A^T\vec{q}^T = \vec{x}A\vec{q}^T \le v, \\ -\vec{p}A^T\vec{q}^T = \vec{p}A\vec{q}^T = v. \end{cases} \tag{1}$$

Then,

$$\begin{cases} \vec{y}A\vec{p}^T = \vec{p}A^T\vec{y}^T \le -v, \\ \vec{q}A\vec{x}^T = \vec{x}A^T\vec{q}^T \ge -v, \\ \vec{q}A\vec{p}^T = \vec{p}A^T\vec{q}^T = -v, \end{cases} \tag{2}$$

that is, $(-v, \vec{q}, \vec{p})$ solves A. Therefore, $v = -v$ which implies $v = 0$.

11. (a)Since

$$\begin{cases} A\vec{y}^T = v\vec{1}^T, \\ \vec{y}A = \vec{y}A^T = v\vec{1}, \\ \vec{y}A\vec{y}^T = v\vec{1}\vec{y}^T = v, \end{cases} \tag{3}$$

by the Minimax Theorem, v is the value of A.
(b)

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

The entry $a_{33} = -1$ is a saddle point, thus the value of A is -1. However,

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

12. (a) If $\lambda_1 \leq 0$ and $\lambda_n > 0$, then the entry $a_{n1} = 0$ is a saddle point and the value of D is 0.
(b) If $\lambda_1 > 0$, then all $\lambda_i \geq \lambda_1 > 0$. By the principle of indifference, solve $v\vec{1} \cdot D^{-1} \cdot \vec{1} = 1$, we obtain

$$v = \frac{1}{\frac{1}{\lambda_1} + ... \frac{1}{\lambda_n}}$$

. In addition,

$$\vec{p} = v\vec{1}D^{-1} = (\frac{\lambda_1}{\frac{1}{\lambda_n} + ... \frac{1}{\lambda_n}}, ..., \frac{\lambda_1}{\frac{1}{\lambda_1} + ... \frac{1}{\lambda_n}})$$

and $\vec{q} = \vec{p}$ since D is symmetric.

13. (a) $x = (1, 1, 2, 3, 5)$ and $a = -2$.
(b) $y = (1, 0, 2, 1, 4)$ and $b = -3$.
(c) $A(\alpha\vec{x} + \beta\vec{y})^T = v\vec{1}^T$ implies $\beta = v$ and $\alpha = -2v$. On the other hand, $(\alpha\vec{x} + \beta\vec{y}) \cdot \vec{1} = 1$ implies $v = -1/16$ and $\alpha = 1/8$, $\beta = -1/16$. Thus, $q = \frac{1}{16}(1, 2, 2, 5, 6)$. Since A is symmetric, then p=q.

14.(a) By the Theorem 2.2.2, the maximin strategy is $(\frac{8k-1}{16k-6}, \frac{8k-5}{16k-6})$, the minimax strategy is $(\frac{8k-2}{16k-6}, \frac{8k-4}{16k-6})$ and the value of the game is $\frac{1}{3-8k}$.
(b) By the principle of indifference, solve $v\vec{1} \cdot D^{-1} \cdot \vec{1} = 1$, we obtain

$$v = \frac{1}{r_1 + ... + r_n}$$

.

(c)

$$A^{-1} = \begin{pmatrix} A_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & A_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & A_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{25}^{-1} \end{pmatrix}.$$

By the principle of indifference, solve $v\vec{1} \cdot A^{-1} \cdot \vec{1} = 1$, we obtain

$$v = \frac{1}{\sum_{k=1}^{k=25}(3 - 8k)}.$$