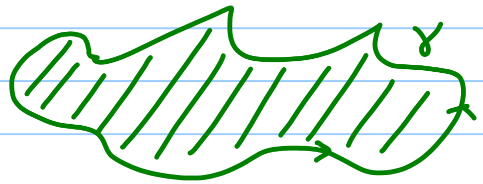
MMAT 5220 Complex Analysis and Its Applications

Lecture 4

§ The Cauchy-Goursat Theorem

Thm (Cauchy-Goursat) If a function f is analytic at all points interior to and on a simple closed contour γ, then

$$\int_\gamma f(z)\,dz = 0$$



Rmk If we assume in addition that $f'(z)$ is continuous at all points interior to and on γ, then the above is just a consequence of Greens Thm and the CR eqns, e.g.

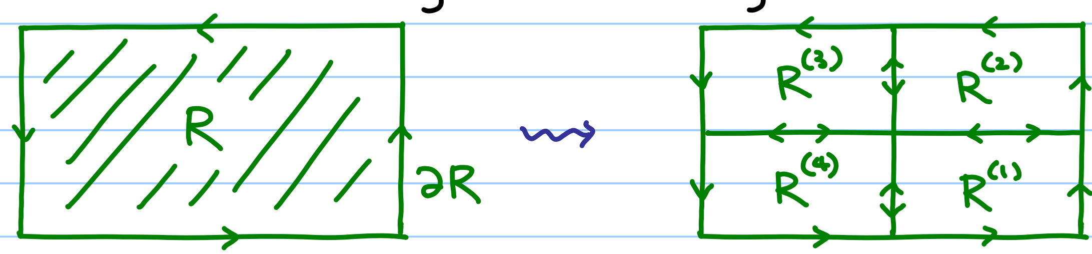
$$\operatorname{Re}\left(\int_\gamma f(z)\,dz\right) = \int_\gamma (u\,dx - v\,dy) = -\int_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

where R denotes the region enclosed by γ.

Lemma 1 If f is analytic on a closed rectangle R (or triangle), then
$$\int_{\partial R} f(z)\,dz = 0$$
where ∂R denotes the boundary of R.

Pf Denote $I(R) := \int_{\partial R} f(z)\,dz$

Subdivides R into 4 congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$:



Then $\int_{\partial R} f(z)\,dz = \int_{\partial R^{(1)}} f(z)\,dz + \int_{\partial R^{(2)}} f(z)\,dz + \int_{\partial R^{(3)}} f(z)\,dz + \int_{\partial R^{(4)}} f(z)\,dz$

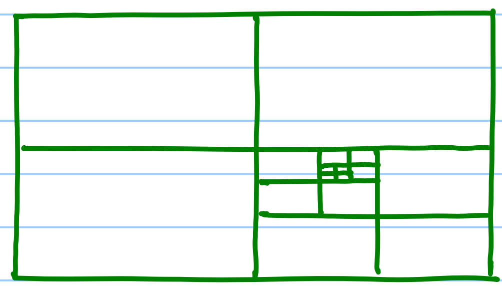
or $I(R) = I(R^{(1)}) + I(R^{(2)}) + I(R^{(3)}) + I(R^{(4)})$

There is an $R^{(k)}$ s.t. $|I(R^{(k)})| \geq \frac{1}{4}|I(R)|$

Denote this $R^{(k)}$ by R_1.

Repeating this process, we obtain a sequence of nested rectangles

$$R =: R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$$



s.t. $\quad |I(R_n)| \geq \frac{1}{4}|I(R_{n-1})| \quad$ for $n = 1, 2, 3, \ldots$

$$\Rightarrow \quad |I(R_n)| \geq \frac{1}{4}|I(R_{n-1})| \geq \frac{1}{4^2}|I(R_{n-2})| \geq \cdots \geq \frac{1}{4^n}|I(R)|$$

By completeness of $\mathbb{R}^2$, R_n converges to a point $z_0 \in R$

i.e. $\forall \delta > 0$, $\exists n_0 > 0$ s.t. $R_n \subset \{z : |z - z_0| < \delta\}$ $\forall n \geq n_0$.

Now f is analytic at z_0, so $\forall \varepsilon > 0$, $\exists \delta > 0$

$$\text{s.t.} \quad \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Observe that

$$\int_{\partial R_n} dz = \int_{\partial R_n} z \, dz = 0 \quad \text{for } n = 1, 2, 3, \ldots$$

So we have $I(R_n) = \int_{\partial R_n} f(z) \, dz = \int_{\partial R_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)] \, dz$

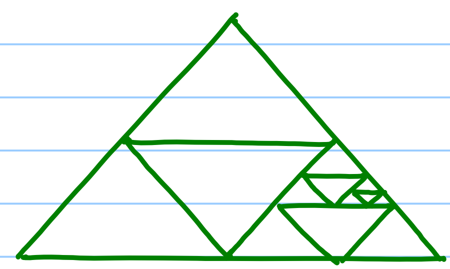
$$\Rightarrow \quad |I(R_n)| \leq \int_{\partial R_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \, dz \leq \varepsilon \int_{\partial R_n} |z - z_0| \, dz$$

If $D :=$ diagonal of R and $L :=$ perimeter of R, then diagonal of $R_n = \frac{1}{2^n} D$ and perimeter of $R_n = \frac{1}{2^n} L$

$$\Rightarrow \quad |I(R_n)| \leq \varepsilon \cdot \frac{D}{2^n} \cdot \frac{L}{2^n} = \frac{\varepsilon}{4^n} DL$$

$$\Rightarrow \quad |I(R)| \leq \varepsilon \cdot DL .$$

Similar argument works for triangles:



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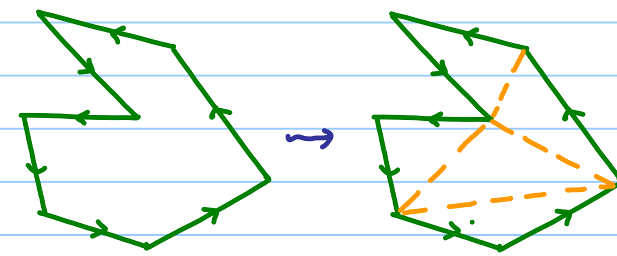
Lemma 2 The Cauchy-Goursat Thm is true for a piecewise linear simple closed contour.

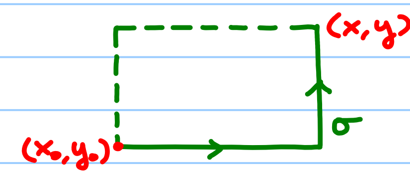
Pf : Subdivide the domain enclosed by γ into triangles. Apply induction and Lemma 1.



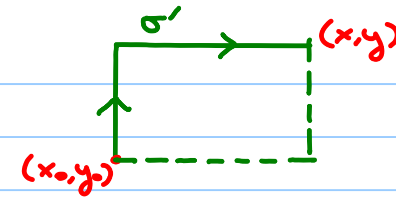
Lemma 3 If $f(z)$ is analytic in an open disk $D = \{ z : |z - z_0| < r \}$, then $\int_\gamma f(z)\, dz = 0$ for any closed contour γ in D.

Sketch of Pf : Fix $z_0 = x_0 + iy_0 \in D$. Then for any $z = x + iy \in D$, consider

1 $\cdots$ $F(z) := \int_\sigma f(z)\,dz = \int_{x_0}^{x} f(t + iy_0)\,dt + i \int_{y_0}^{y} f(x + is)\,ds$



By Lemma 1, we have



② $\cdots$ $$F(z) = \int_{\sigma'} f(z)\,dz = \int_{x_0}^{x} f(t+iy)\,dt + i\int_{y_0}^{y} f(x_0+is)\,ds$$

So $$\frac{\partial F}{\partial x} \overset{②}{=} \lim_{a\to 0} \frac{1}{a} \int_{x}^{x+a} f(t+iy)\,dt = f(x+iy)$$

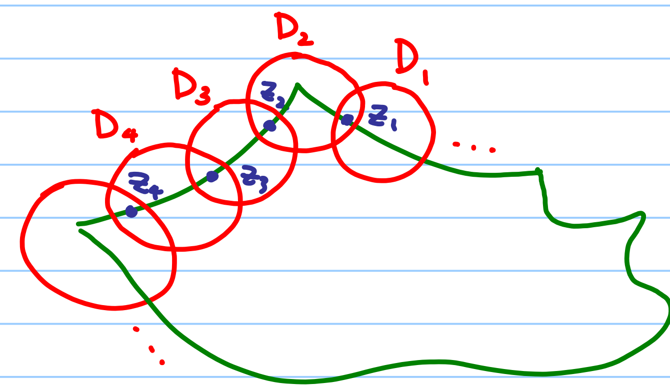
$$\frac{\partial F}{\partial y} \overset{①}{=} \lim_{a\to 0} \frac{i}{a} \int_{y}^{y+a} f(x+is)\,ds = i f(x+iy)$$

Hence $F'(z) = f(z)$. #

Pf of the Cauchy-Goursat Thm

Idea: Using compactness of γ, we cover it by a finite number of open disks $D_1, D_2, \dots, D_N$

s.t. $\begin{cases} \bullet \ \exists\, z_i \in \gamma \cap (D_i \cap D_{i+1}) \ \text{ for } \ i = \ , \dots, N \ (\text{where } D_{N+1} = D_1 \\ \bullet \ \text{The part } \gamma_i \text{ of } \gamma \text{ from } z_{i-1} \text{ to } z_i \text{ sits inside } D_i \\ \bullet \ f \text{ is analytic on } D_i \end{cases}$



Then take the line segment l_i from z_i to z_{i+1} $\forall i$

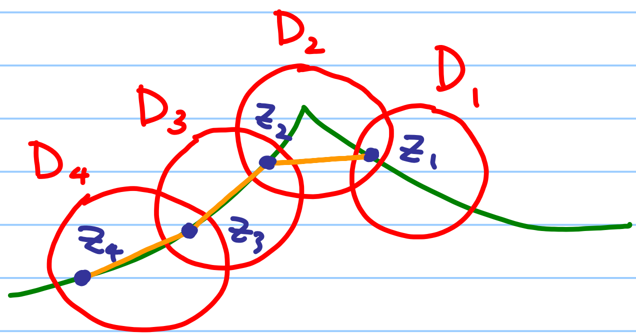
and set $\tilde{\gamma} := \sum_i l_i$

By Lemma 3,

$$\int_{\gamma_i} f(z)\,dz = \int_{\ell_i} f(z)\,dz$$

$$\Rightarrow \quad \int_{\gamma} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz = 0$$

by Lemma 2. #



Def A domain $D \subset \mathbb{C}$ is called **simply connected** if every simple closed curve (or loop) in D encloses only points in D.

Rmk Alternatively, D is simply connected if every loop in D contracts to a pt in D. Intuitively, this means D has "no holes".

Thm (Extended Cauchy-Goursat)

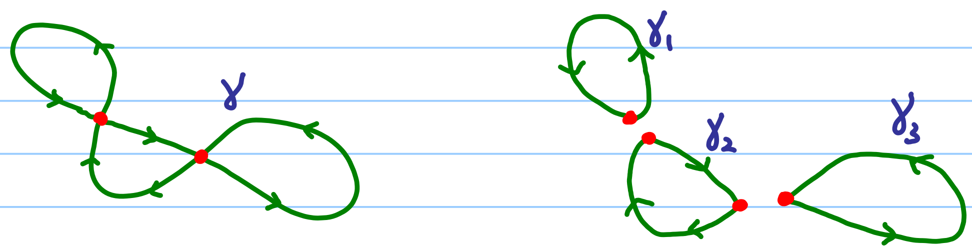
If $f(z)$ is analytic throughout a simply connected domain D, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed contour (not necessarily simple) γ in D

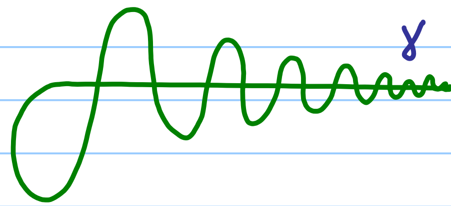
Pf: Let $\gamma \subset D$ be a nonsimple closed contour.

Case 1: γ intersects itself a finite number of times.



Then $\int_\gamma f(z)\,dz = \sum_i \int_{\gamma_i} f(z)\,dz = 0.$

Case 2 : γ intersects itself an infinite number of times



Then we can replace γ by a polygonal contour $\tilde{\gamma}$ (as in the proof of the Cauchy-Goursat Thm).

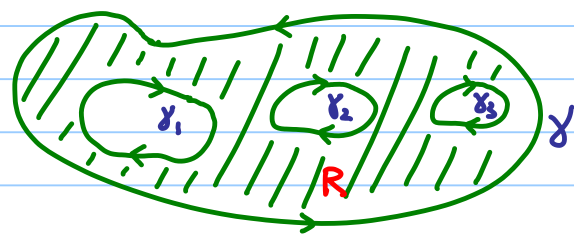
Now $\tilde{\gamma}$ has only a finite number of self-intersections, so by Case 1, we have

$$\int_{\gamma} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz = 0. \quad \#$$

Cor A function which is analytic throughout a simply connected domain D must have an antiderivative in D.

Cor Entire functions have antiderivatives.

Cor Let γ and γ_k $(k=1,2,\dots,n)$ be simple closed contours as follows:

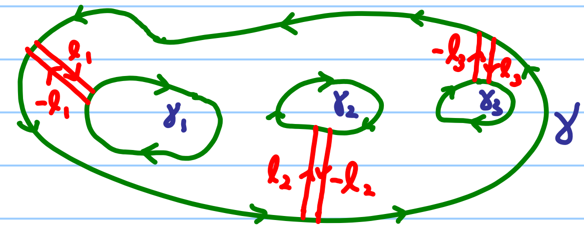


$$\partial R = \gamma + \gamma_1 + \gamma_2 + \cdots - \gamma_n$$

If f is analytic on $\overline{R} := R \cup \partial R$, then

$$\int_\gamma f(z)\,dz + \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz = 0$$

Pf : Cut R as shown in the picture.



Then $\mu :=$ "γ" $+ (\ell_1 + \gamma_1 - \ell_1) + (\ell_2 + \gamma_2 - \ell_2)$

$+ \cdots + (\ell_n + \gamma_n - \ell_n)$

is a simple closed contour. #

§ Cauchy integral formulas

Thm Let f be analytic at all points interior to and on a simple closed contour γ in positive orientation. Then for any $z_0 \in \mathbb{C}$ interior to γ, we have the Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz .$$

Thm Let f be analytic at all points interior to and on a simple closed contour γ in positive orientation. Then for any $z \in \mathbb{C}$ interior to γ and every $n \in \mathbb{N}$, we have the Cauchy integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+1}}\, ds .$$

Cor If $f(z) = u(x,y) + iv(x,y)$ is analytic at a pt $z_0 = x_0 + iy_0$, then f is infinitely (complex) differentiable at z_0. In particular, u and v are infinitely differentiable at (x_0, y_0).

Rmk Note again the sharp contrast with functions of two variables which may not even have continuous partial derivatives even if differentiable at a point.