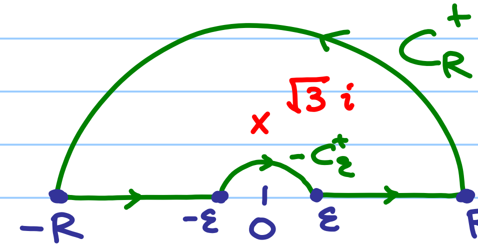
MMAT 5220 Complex Analysis and Its Applications

Lecture 10

§ Applications of residues (cont'd)

$\boxed{\text{Indented contours}}$

Example 5 (Dirichlet's Integral) Show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$



Sol: Consider the integral of $f(z) = \frac{e^{iz}}{z}$ on Γ:

Cauchy's Residue Theorem

$$\Rightarrow \quad 0 = \int_{C_R^+} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx - \int_{C_\varepsilon^+} \frac{e^{iz}}{z}\,dz + \int_\varepsilon^R \frac{e^{ix}}{x}\,dx$$

But $\displaystyle\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx = -\int_\varepsilon^R \frac{e^{-ix}}{x}\,dx \Rightarrow \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_\varepsilon^R \frac{e^{ix}}{x}\,dx = 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx$

On the other hand,

$$\left|\int_{C_R^+} \frac{e^{iz}}{z}\,dz\right| \leq \int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R} \to 0 \quad \text{as} \quad R\to\infty$$

Finally,

$$\int_{C_\varepsilon^+} \frac{e^{iz}}{z}\,dz = \int_{C_\varepsilon^+} \frac{1}{z}\left(1+iz+\frac{(iz)^2}{2}+\cdots\right)dz$$

$$= \int_{C_\varepsilon^+} \frac{dz}{z} + \underbrace{\int_{C_\varepsilon^+}\left(i+\frac{i^2}{2}z+\cdots\right)dz}_{\text{tends to } 0 \text{ as } \varepsilon\to 0}$$

$$\to \pi i \quad \text{as } \varepsilon \to 0$$

Hence, letting $R\to\infty$ and $\varepsilon\to 0$ gives

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Rmk In general, if $f(z)$ has a simple pole at z_0, then

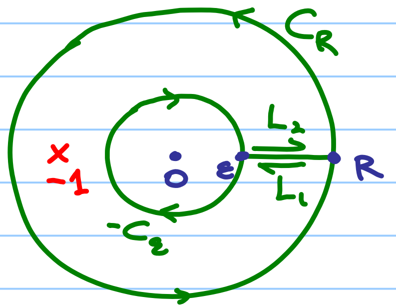
$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon^+} f(z)\,dz = \pi i \operatorname*{Res}_{z=z_0} f(z).$$

semicircle centered at z_0

<u>Example 6</u> Evaluate $\int_0^\infty \frac{x^{-a}}{1+x}\,dx$ for $0<a<1$.

<u>Sol</u>: Consider $f(z) = \frac{z^{-a}}{1+z}$ where the branch of $z^{-a} = e^{-a\log z}$

is given by $\log z = \log r + i\theta$, $0<\theta<2\pi$.



Consider the contour $\Gamma = C_R + L_1 - C_\varepsilon + L_2$

We have $\log z = \begin{cases} \log r + 2\pi i & \text{on } L_1 \\ \log r & \text{on } L_2 \end{cases}$

$$\Rightarrow \int_{L_1} f(z)\,dz = \int_R^\varepsilon \frac{e^{-a(\log r + 2\pi i)}}{1+r}\,dr = -e^{2a\pi i}\int_\varepsilon^R \frac{r^{-a}}{1+r}\,dr$$

and $\int_{L_2} f(z)\,dz = \int_\varepsilon^R \frac{r^{-a}}{1+r}\,dr$

On the other hand,

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| = \left| \int_{C_\varepsilon} \frac{z^{-a}}{1+z}\,dz \right| \leqslant \frac{\varepsilon^{-a}}{1-\varepsilon} \cdot 2\pi\varepsilon = \frac{2\pi\varepsilon^{1-a}}{1-\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\left| \int_{C_R} f(z)\,dz \right| = \left| \int_{C_R} \frac{z^{-a}}{1+z}\,dz \right| \leqslant \frac{R^{-a}}{R-1} \cdot 2\pi R \longrightarrow 0 \quad \text{as} \quad R \to \infty$$

Hence, Cauchy integral formula implies that

$$\left(1 - e^{-2a\pi i}\right) \int_0^\infty \frac{x^{-a}}{1+x}\,dx = 2\pi i \operatorname*{Res}_{z=-1} \left(\frac{z^{-a}}{1+z} \right) = 2\pi i\, e^{-a\pi i}.$$

$$\Rightarrow \int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{2\pi i\, e^{-a\pi i}}{1 - e^{-2a\pi i}} = \frac{\pi}{\sin a\pi}. \quad \#$$

Trigonometric integrals

For integrals of the form $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$, we consider

$$\int_{|z|=1} F\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2} \right) \frac{dz}{iz}$$

<u>Example 7</u> Evaluate $I = \int_0^{\pi} \frac{d\theta}{2-\cos\theta}$

<u>Sol</u>: Note that $\int_{\pi}^{2\pi} \frac{d\theta}{2-\cos\theta} = \int_0^{\pi} \frac{d\theta}{2-\cos\theta}$.

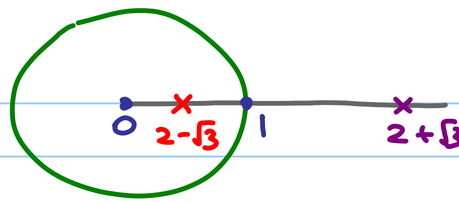
So $I = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{2-\cos\theta}$

$$= \frac{1}{2} \int_{|z|=1} \frac{1}{2 - \frac{z+z^{-1}}{2}} \frac{dz}{iz}$$

$$= i \int_{|z|=1} \frac{dz}{z^2 - 4z + 1}$$

$$= i \cdot 2\pi i \operatorname*{Res}_{z=2-\sqrt{3}} \left(\frac{1}{z^2-4z+1} \right)$$

$$= \frac{\pi}{\sqrt{3}} . \quad \#$$



§ Winding numbers

Def A function $f: D \to \mathbb{C}$ is called meromorphic if the only singular points of f in D are poles.

Thm Let γ be a positively oriented simple closed contour. Let f be a function meromorphic inside and on γ such that f has no zeros or poles on γ. Let $a_1, \dots, a_n$ be zeros of f inside γ of orders $\alpha_1, \dots, \alpha_n$ respectively, and $b_1, \dots, b_m$ be poles of f inside γ of orders $\beta_1, \dots, \beta_m$ respectively. Then

$$\frac{1}{2\pi i} \int_\gamma \varphi(z) \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} \alpha_k \varphi(a_k) - \sum_{j=1}^{m} \beta_j \varphi(b_j)$$

for any function φ which is analytic inside and on γ.

Pf: By Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_\gamma \varphi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k}\left(\varphi(z)\frac{f'(z)}{f(z)}\right) + \sum_{j=1}^{m} \operatorname*{Res}_{z=b_j}\left(\varphi(z)\frac{f'(z)}{f(z)}\right)$$

Since zeros are isolated, for each $k=1,\dots,n$, we can write

$$f(z) = (z-a_k)^{\alpha_k} g(z) \quad \text{in } B(a_k,\varepsilon)$$

for some $\varepsilon>0$ and for some analytic g with $g(a_k)\neq 0$.

$$\Longrightarrow \varphi(z)\frac{f'(z)}{f(z)} = \frac{\alpha_k\varphi(z)}{z-a_k} + \varphi(z)\frac{g'(z)}{g(z)} \quad \text{in } B(a_k,\varepsilon)$$

But $\varphi(z)\frac{g'(z)}{g(z)}$ is analytic, so

$$\operatorname*{Res}_{z=a_k}\left(\varphi(z)\frac{f'(z)}{f(z)}\right) = \alpha_k\varphi(a_k).$$

Similarly, poles are all isolated and for each $j=1,\dots,m$, we have

$$f(z) = \frac{h(z)}{(z-b_j)^{\beta_j}} \quad \text{in } B(b_j, \varepsilon)$$

for some $\varepsilon > 0$ and for some analytic h with $h(b_j) \neq 0$.

$$\Longrightarrow \varphi(z)\frac{f'(z)}{f(z)} = -\frac{\beta_j \varphi(z)}{z-b_j} + \varphi(z)\frac{h'(z)}{h(z)} \quad \text{in } B(b_j, \varepsilon)$$

But $\varphi(z)\frac{h'(z)}{h(z)}$ is analytic, so

$$\operatorname*{Res}_{z=a_k}\left(\varphi(z)\frac{f'(z)}{f(z)}\right) = -\beta_j \varphi(b_j). \quad \#$$

Cor (Argument Principle)

Let γ be a positively oriented simple closed contour. Let f be a function meromorphic inside and on γ such that f has no zeros or poles on γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N_0(f) - N_\infty(f)$$

where $N_0(f)$ and $N_\infty(f)$ are the numbers of zeros and poles of f (counted with multiplicities/orders) inside γ respectively.

Def Let $z_0 \in \mathbb{C}$ and γ be a closed contour (not necessarily simple) such that $z_0 \notin \gamma$. Then the integer

$$n(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$$

is called the winding number of γ around z_0.

Rmks • $n(\gamma, z_0)$ measures the "number of turns" γ made around z_0

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_\gamma d \log(\gamma(t) - z_0) = \frac{1}{2\pi} \Delta_\gamma \arg(\gamma(t) - z_0) \in \mathbb{Z}$$

• By the change of variable $w = f(z)$, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = n(f(\gamma), 0)$$

So the Argument Principle can be written as

$$n(f(\gamma), 0) = N_0(f) - N_\infty(f).$$

e.g. • For $\gamma = \{z \in \mathbb{C} : |z| = 1\}$, +ve'ly oriented, $n(\gamma, 0) = 1$, and and if $f(z) = z^m$ $(m \in \mathbb{Z})$, then $n(f(\gamma), 0) = m$.

• Let $f(z) = \dfrac{z^3(z-8)^2}{(z-5)^4(z+2)^2(z-1)^5}$ and $\gamma = \{z \in \mathbb{C} : |z| = 4\}$, +ve'ly oriented.

zeros of f inside γ : $z = 0$ (order 3)

poles of f inside γ : $z = 1$ (order 5) and $z = -2$ (order 2)

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = 3 - (5+2) = -4$$

§ Rouché's Theorem

Thm (Rouché's Theorem)
If f and g are analytic on and inside a simple closed contour γ s.t. $|g(z)| < |f(z)|$ $\forall z \in \gamma$, then $N_0(f) = N_0(f+g)$ inside γ.

Pf: Consider the meromorphic function $F := \frac{f+g}{f}$.

The assumption implies that $|F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1$ on γ

$\Rightarrow$ 0 is outside $F(\gamma)$ $\Rightarrow$ $n(F(\gamma), 0) = 0$.

Hence $$0 = \frac{1}{2\pi i}\int_\gamma \frac{F'}{F}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz$$
$$= N_0(f+g) - N_0(f). \quad \#$$

e.g. ① All the zeros of $h(z) = z^5 + z + 3$ lie inside $|z| < 2$.

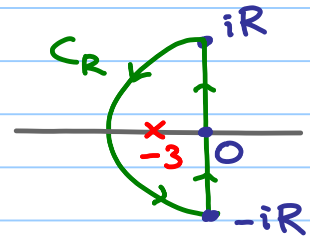
Sol : Let $f(z) = z^5$ and $g(z) = z + 3$. Then both f & g are entire.

Also $|g(z)| = |z+3| \leq |z| + 3 = 5 < 2^5 = |f(z)|$ on $|z| = 2$.

By Rouché's Thm, $h = f + g$ and f have the same number of zeros inside $|z| = 2$. But h has exactly 5 zeros.

So all zeros of h lie inside $|z| < 2$.

(2) The equation $z+3+2e^z=0$ has precisely one solution in the left half-plane.



Sol: Let $f(z)=z+3$ and $g(z)=2e^z$.

Consider the contour γ:

For $R>0$ sufficiently large, we have $|f(z)|\geq 3$ on γ, while $|g(z)|=2|e^z|=2e^{\operatorname{Re} z}\leq 2e^0=2$, so $|g(z)|<|f(z)|$ on γ.

Hence $f+g=z+3+2e^z$ has the same number of zeros inside γ as $f=z+3$, i.e. one.

Letting $R\to+\infty$, we see that $z+3+2e^z=0$ has exactly one solution in the left half-plane.