MMAT 5220 Complex Analysis and Its Applications

Lecture 1

§ The complex plane

Notations :

$\mathbb{N}$ = set of natural numbers $= \{0, 1, 2, 3, \dots\}$

$\mathbb{Z}$ = set of integers $= \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Q}$ = set of rational numbers

$\mathbb{R}$ = set of real numbers

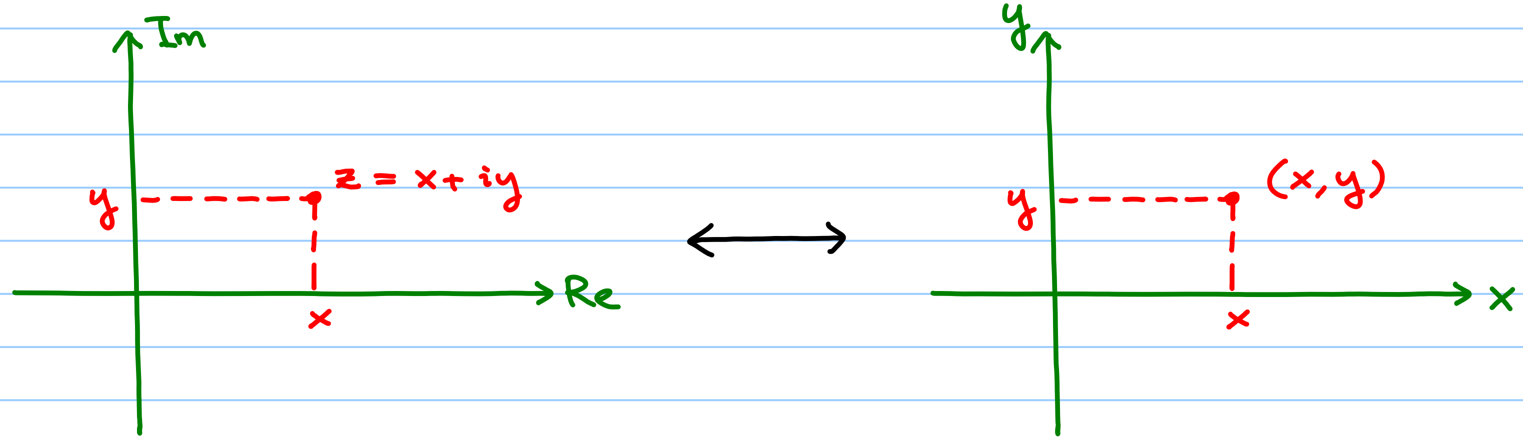
$\mathbb{C}$ = set of complex numbers

$= \{x + iy : x, y \in \mathbb{R}\}$ where $i = \sqrt{-1}$

Given $z = x + iy \in \mathbb{C}$, $\operatorname{Re} z := x$ is called the real part of z
and $\operatorname{Im} z := y$ is called the imaginary part of z

$\rightsquigarrow$ one-to-one correspondence

$$\mathbb{C} \cong \mathbb{R}^2$$



This is why we call $\mathbb{C}$ the complex plane.

§ Algebraic operations on $\mathbb{C}$

Given $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}$, we define

1) (Addition) $\quad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2) (Multiplication) $\quad z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Not hard to check

commutativity	$z_1 + z_2 = z_2 + z_1$	$z_1z_2 = z_2z_1$
associativity	$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	$(z_1z_2)z_3 = z_1(z_2z_3)$
exist. of id.	$z + 0 = z \quad \forall z \in \mathbb{C}$	$z \cdot 1 = z \quad \forall z \in \mathbb{C}$
exist. of inv.	$\forall z \in \mathbb{C}, \exists -z \in \mathbb{C}$ s.t. $z + (-z) = 0$	$\forall z \in \mathbb{C}\setminus\{0\}, \exists z^{-1} \in \mathbb{C}$ s.t. $z \cdot z^{-1} = 1$

If $z = x + iy$ then $z^{-1} = \dfrac{x - iy}{x^2 + y^2}$

distributive law & $z_1 \cdot (z_2 + z_3) = z_1z_2 + z_1z_3$

$\rightsquigarrow (\mathbb{C}, +, \cdot)$ is a field (so is $\mathbb{Q}$ and $\mathbb{R}$).

Thm (Fundamental Theorem of Algebra)
Every complex polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ (where $a_i \in \mathbb{C}\ \forall i$) of degree $n \geq 1$ has a zero in $\mathbb{C}$, i.e., $\exists\, z_0 \in \mathbb{C}$ s.t. $f(z_0) = 0$.

Will be proved later in this course!

§ Further operations

Def The modulus (or absolute value) of $z = x + iy \in \mathbb{C}$ is defined as

$$|z| := \sqrt{x^2 + y^2} \in \mathbb{R}_{\geq 0}$$

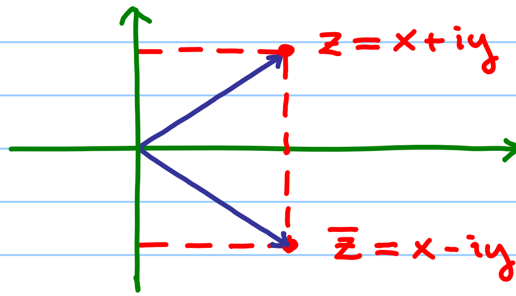
= length of the vector $(x, y) \in \mathbb{R}^2$

= distance between (x, y) and $(0, 0)$

Def The complex conjugate (or simply conjugate) of $z = x + iy \in \mathbb{C}$ is defined as

$$\bar{z} := x - iy \in \mathbb{C}$$

(= reflection of z along the real axis)



Basic properties:

1) $\bar{\bar{z}} = z$

2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$,
$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$,
$\overline{z^{-1}} = \bar{z}^{-1}$

3) $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$

4) $|z| = |\bar{z}|$

5) $|z|^2 = z\bar{z}$

6) $|z_1 z_2| = |z_1||z_2|$

7) $\begin{cases} \operatorname{Re} z \leqslant |\operatorname{Re} z| \leqslant |z| \\ \operatorname{Im} z \leqslant |\operatorname{Im} z| \leqslant |z| \end{cases}$

8) (Triangle Inequality)

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

and "=" iff $z_1 = k z_2$ for some $k \in \mathbb{R}$ (i.e., $z_1 \parallel z_2$)

<u>Pf</u>:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_2 \\ &= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \\ &\leqslant |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2. \end{aligned}$$

$$\begin{aligned} \text{Equality holds} &\iff \mathrm{Re}(z_1\bar{z}_2) = |z_1\bar{z}_2| \\ &\iff z_1\bar{z}_2 \in \mathbb{R} \\ &\iff z_1/z_2 \in \mathbb{R}. \end{aligned}$$

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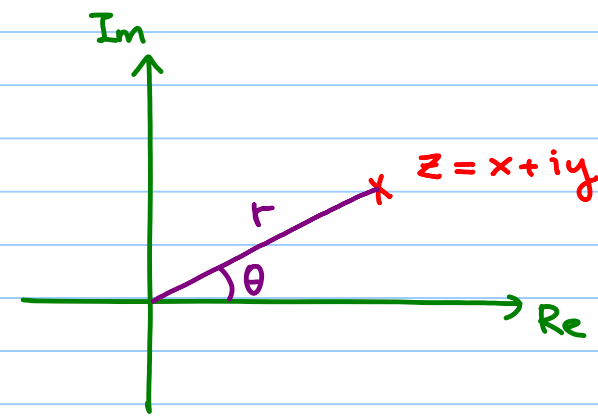
§ Polar coordinates

Cartesian coordinates $\longleftrightarrow$ Polar coordinates

(x, y) $\qquad$ (r, θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



$\rightsquigarrow \quad z = x + iy = r(\cos\theta + i\sin\theta)$

- $r = |z|$
- θ is undefined for $z = 0$
- For $z \neq 0$, θ is defined only up to $2k\pi$ for $k \in \mathbb{Z}$; each value of θ s.t. $z = |z|(\cos\theta + i\sin\theta)$ is called an argument of z.

- We set $\arg z :=$ set of all arguments of $z \in \mathbb{C} \setminus \{0\}$.
- The principal argument of z, denoted as $\operatorname{Arg} z$, is the argument of z lying in $(-\pi, \pi]$, i.e., $-\pi < \operatorname{Arg} z \leq \pi$.

So we have $\arg z = \{ \operatorname{Arg} z + 2k\pi : k \in \mathbb{Z} \}$

Euler's formula

$$e^{i\theta} := \cos\theta + i \sin\theta$$

(justification : by Taylor series

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i \sin\theta$$)

So now we have $z = re^{i\theta}$, and

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$ by compound angle formula

$\Rightarrow$ (de Moivre's Thm) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ as $(e^{i\theta})^n = e^{in\theta}$

For $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2} \in \mathbb{C}$, we have

$$\begin{cases} z_1z_2 = r_1r_2\,e^{i(\theta_1+\theta_2)} \\ \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)} \end{cases} \quad \text{or} \quad \begin{cases} |z_1z_2^{\pm1}| = |z_1|\,|z_2|^{\pm1} \\ \arg(z_1z_2^{\pm1}) = \arg z_1 \pm \arg z_2 \end{cases}$$

Note : Polar coordinates work well for $\times, \div$
while Cartesian coordinates work well for $+, -$

§ Elementary functions on $\mathbb{C}$

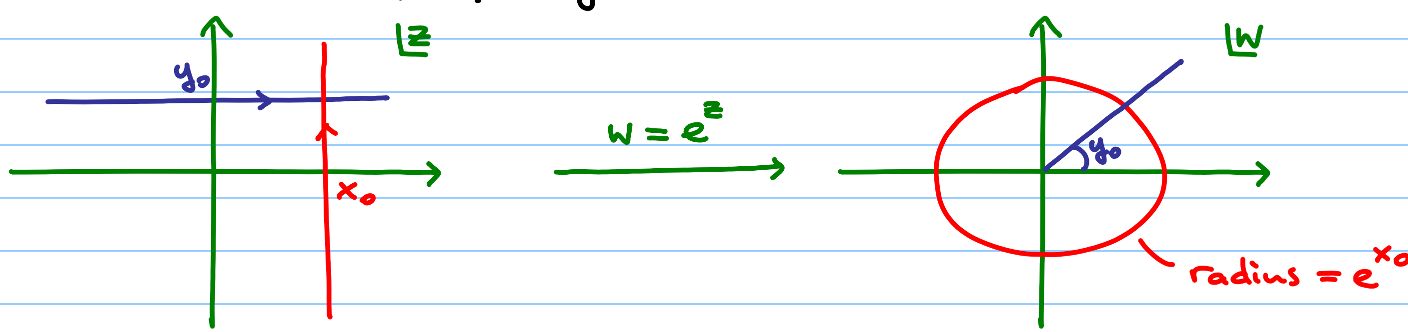
- Polynomial functions

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

- Exponential function

$$e^z = e^{x+iy} := e^x \cdot e^{iy} \quad \text{for } z = x+iy \in \mathbb{C}$$

It satisfies the property $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$



- Trigonometric functions

$$\cos z := \frac{e^{iz}+e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz}-e^{-iz}}{2i}$$

basic properties:

1) $\cos(-z) = \cos z, \quad \sin(-z) = -\sin z \qquad \forall z \in \mathbb{C}$

2) $\cos(z+2\pi) = \cos z, \quad \sin(z+2\pi) = \sin z \qquad \forall z \in \mathbb{C}$

3) $\cos^2 z + \sin^2 z = 1 \qquad \forall z \in \mathbb{C}$

4) $\begin{cases} \cos(z+w) = \cos z \cos w - \sin z \sin z \\ \sin(z+w) = \sin z \cos w + \cos z \sin w \end{cases} \qquad \forall z, w \in \mathbb{C}$

- Logarithm

$$\log z := \log |z| + i \arg z$$

(justification : $\log(re^{i\theta}) = \log r + i\theta$)

But $\arg z$ is multi-valued, so to define an honest function, we need to choose a branch of log, namely, we fix $\alpha \in \mathbb{R}$ and set $\log z := \log |z| + i \arg z$ with $\alpha < \arg z \leq \alpha + 2\pi$

The principal branch of log, denoted $\operatorname{Log} z$, is given by

$$\operatorname{Log} z := \log |z| + i \operatorname{Arg} z \quad (\text{i.e., with } \alpha = -\pi)$$

- Power functions

Given $c \in \mathbb{C}$, we can define the (multi-valued) power function

$$z^c := e^{c \log z}$$

Each branch of $\log z$ defines a branch of z^c.
In particular, the principal branch of z^c, also called the principal value of z^c, is given by

$$\text{P.V. } z^c := e^{c \operatorname{Log} z}$$

e.g. The square root function is defined by $\sqrt{z} = z^{\frac{1}{2}} = e^{\frac{1}{2}\log z}$

$$= \sqrt{|z|}\, e^{\frac{i}{2}\arg z}$$