**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MMAT5220 Complex Analysis and Its Applications 2019-20**
**Homework 1**
**Due Date: 20th February 2020**

## Compulsory Part

1. Let $n \geq 1$.

   (a) Show that $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ if $z \neq 1$.

   (b) Use part (a) to deduce **Lagrange's trigonometric identity**:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

   when $\theta$ is not a multiple of $2\pi$.

**Solution.**

(a) Note that

$$\begin{aligned}(1-z)(1+z+z^2+\cdots+z^n) &= 1+z+z^2+\cdots+z^n-(z+z^2+z^3+\cdots+z^{n+1})\\ &= 1-z^{n+1}\end{aligned}$$

When $z \neq 1$, we divide both sides by $1-z$ to obtain the identity.

(b) Let $z = e^{i\theta}$, by part (a), when $e^{i\theta} \neq 1$, i.e. when $\theta$ is not a multiple of $2\pi$, we have

$$1 + e^{i\theta} + (e^{i\theta})^2 + \cdots + (e^{i\theta})^n = \frac{1-(e^{i\theta})^{n+1}}{1-e^{i\theta}} \tag{1}$$

Now, notice that $\cos(k\theta) = \mathrm{Re}(e^{ik\theta}) = \mathrm{Re}[(e^{i\theta})^k]$. Therefore, the real part of Equation (1) is $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$. Moreover, the RHS is

$$\begin{aligned}\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} &= \frac{e^{-\frac{i\theta}{2}}(1-e^{i(n+1)\theta})}{e^{-\frac{i\theta}{2}}(1-e^{i\theta})} = \frac{e^{-\frac{i\theta}{2}}-e^{i(n+\frac{1}{2})\theta}}{e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}}\\ &= \left(\frac{e^{-\frac{i\theta}{2}}-e^{i(n+\frac{1}{2})\theta}}{2i}\right) \Bigg/ \left(\frac{e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}}{2i}\right)\\ &= -\frac{1}{\sin\frac{\theta}{2}}\frac{e^{-\frac{i\theta}{2}}-e^{i(n+\frac{1}{2})\theta}}{2i}\\ &= \frac{i}{2\sin\frac{\theta}{2}}(e^{\frac{-i\theta}{2}}-e^{i(n+\frac{1}{2})\theta})\\ &= \frac{i}{2\sin\frac{\theta}{2}}(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}-\cos(n+\frac{1}{2})\theta-i\sin(n+\frac{1}{2})\theta)\end{aligned}$$

Its real part is $\frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$. This shows the required identity. ◀

2. Show that $|z_1 - z_2| \geq ||z_1| - |z_2||$ for any $z_1, z_2 \in \mathbb{C}$.

**Solution.** Note that

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(\overline{z_1}z_2) \\ &\geq |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2| \\ &= ||z_1| - |z_2||^2 \end{aligned}$$

The result follows by taking square root on both sides. Notice that in calculation, we applied $\mathrm{Re}(\overline{z_1}z_2) \leq |\overline{z_1}|\,|z_2| = |z_1|\,|z_2|$. ◀

3. Consider the function

$$T(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. Show that

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

**Solution.**

(a) If $c = 0$, then by assumption $ad - bc \neq 0$, we have $a, d \neq 0$. To see that $\lim_{z\to\infty} T(z) = \infty$, we only need to check that $\lim_{w\to 0} \frac{1}{T(\frac{1}{w})} = 0$. Note that $\frac{1}{T(\frac{1}{w})} = \frac{dw}{bw+a}$, and $b(0) + a = a \neq 0$. Therefore, $\lim_{w\to 0} \frac{1}{T(\frac{1}{w})} = 0$.

(b) $\lim_{z\to\infty} T(z) = \lim_{w\to 0} T(\frac{1}{w}) = \lim_{w\to 0} \frac{bw+a}{dw+c} = \frac{a}{c}$, because $c \neq 0$.

$\lim_{z\to -d/c} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = \frac{c(-\frac{d}{c})+d}{a(-\frac{d}{c})+b} = 0$, because $a(-\frac{d}{c}) + b = \frac{bc-ad}{c} \neq 0$. Therefore, we conclude that $\lim_{z\to -d/c} T(z) = \infty$.

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4. For the following functions defined on the whole complex plane, show that they are complex differentiable at every point by computing the partial derivatives of their real and imaginary parts and verifying the Cauchy-Riemann equations:

(a) $f(z) = z^2$.

(b) $f(z) = e^z$.

(Remark: Functions which are complex differentiable on the whole complex plane are called **entire functions**.)

**Solution.** (a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. Then,

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy).$$

The real part of $f$ is $u(x, y) = x^2 - y^2$ and the imaginary part of $f$ is $v(x, y) = 2xy$. Notice that the partial derivatives of $u, v$ are

$$\begin{aligned} u_x &= 2x & \qquad u_y &= -2y \\ v_x &= 2y & \qquad v_y &= 2x \end{aligned}$$

Therefore, $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. That is, $f$ satisfies the Cauchy-Riemann equations. Moreover, since all partial derivatives are continuous, $f$ is complex differentiable.

(b) $f(z) = e^{x+iy} = e^x \cos y + ie^x \sin y$. That is, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Note that

$$\begin{aligned} u_x &= e^x \cos y & \qquad u_y &= -e^x \sin y \\ v_x &= e^x \sin y & \qquad v_y &= e^x \cos y \end{aligned}$$

Therefore, $f$ satisfies the Cauchy-Riemann equations $u_x = v_x$ and $u_y = -v_x$. Since all partial derivatives are continuous, $f$ is complex differentiable.

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5. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \bar{z}$. By considering the Cauchy-Riemann equations, show that $f'(z)$ does not exist at any point.

**Solution.** Note that the real part of function $f$ is $u(x, y) = x$ and the imaginary part is $v(x, y) = -y$. To check the Cauchy-Riemann equations, we need to calculate the partial derivatives:

$$\begin{aligned} u_x &= 1 & \qquad u_y &= 0 \\ v_x &= 0 & \qquad v_y &= -1 \end{aligned}$$

Therefore, one of the Cauchy-Riemann equations $u_x = v_y$ fails for any point $(x_0, y_0)$ in the complex plane, hence $f'(z)$ does not exist at any point.

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## Optional Part

1. Show, by definition, that

   (a) $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for any $z_1, z_2 \in \mathbb{C}$;

   (b) $\log(z_1 z_2) = \log z_1 + \log z_2$ for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

   (c) $\sin^2 z + \cos^2 z = 1$ for any $z \in \mathbb{C}$.

**Solution.** (a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then,

$$\begin{aligned}
e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_1+y_2)} \\
&= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\
&= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i \sin y_1 \cos y_2 + i \cos y_1 \sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\
&= e^{z_1} \cdot e^{z_2}
\end{aligned}$$

(b) Once you chose a branch for the $\log$ function, you can find some $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ such that

$$\log(z_1 z_2) \neq \log z_1 + \log z_2.$$

For example, consider the principal branch $-\pi < \operatorname{Arg} z \leq \pi$. For $z_1 = z_2 = -1$, we have $\log z_1 = \log z_2 = i\pi$, but $\log(z_1 z_2) = 0 \neq \log z_1 + \log z_2$.

On the other hand, if we first fix $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, then we can always choose a branch of $\log$ such that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{2}$$

For our discussion, we may clarify some terminologies. Let $z = re^{i\theta}$ with $\theta \in (0, 2\pi]$. We say that $z$ belongs to

| | | | |
|---|---|---|---|
| quadrant I | if $0 < \theta \leq \frac{\pi}{2}$; | quadrant II | if $\frac{\pi}{2} < \theta \leq \pi$; |
| quadrant III | if $\pi < \theta \leq \frac{3\pi}{2}$; | quadrant IV | if $\frac{3\pi}{2} < \theta \leq 2\pi$. |

If we chose the principal branch $-\pi < \arg z \leq \pi$ for the $\log$ function, then Equation (2) holds when $z_1$ is in quadrants I and II, together with $z_2$ in quadrants III and IV. To verify it, note that

$$\begin{aligned}
\log z_1 &= \log|z_1| + i \arg z_1 \quad \text{with } 0 < \arg z_1 \leq \pi, \\
\log z_2 &= \log|z_2| + i \arg z_2 \quad \text{with } -\pi < \arg z_2 \leq 0.
\end{aligned}$$

Hence,

$$z_1 z_2 = |z_1| e^{i \arg z_1} |z_2| e^{i \arg z_2} = |z_1 z_2| e^{i(\arg z_1 + \arg z_2)}$$

with $-\pi < \arg z_1 + \arg z_2 \leq \pi$, which is in our chosen branch. Therefore, Equation (2) holds. Similarly, the table below shows for different branches of $\log$ function, when $z_1, z_2$ will satisfy Equation (2).

| $z_1$ \ $z_2$ | I | II | III | IV |
|---|---|---|---|---|
| I | ✓ | × | ✓ | ✓ |
| II | × | × | ✓ | ✓ |
| III | ✓ | ✓ | × | × |
| IV | ✓ | ✓ | × | ✓ |

(a) Branch: $-\pi < \arg z \leq \pi$

| $z_1$ \ $z_2$ | I | II | III | IV |
|---|---|---|---|---|
| I | ✓ | ✓ | ✓ | × |
| II | ✓ | ✓ | × | × |
| III | ✓ | × | × | × |
| IV | × | × | × | × |

(b) Branch: $0 < \arg z \leq 2\pi$

| $z_1 \backslash z_2$ | I | II | III | IV |
|---|---|---|---|---|
| I | ✓ | ✓ | × | ✓ |
| II | ✓ | × | × | ✓ |
| III | × | × | × | ✓ |
| IV | ✓ | ✓ | ✓ | × |

(c) Branch: $-\frac{\pi}{2} < \arg z \leq \frac{3\pi}{2}$

| $z_1 \backslash z_2$ | I | II | III | IV |
|---|---|---|---|---|
| I | × | × | × | × |
| II | × | × | × | × |
| III | × | × | ✓ | × |
| IV | × | × | × | × |

(d) Branch: $\pi < \arg z \leq 3\pi$

Therefore, for any nonzero fixed $z_1, z_2$, we can choose a proper branch of $\log$ function such that Equation (2) holds.

(c) Note that

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} + e^{-2iz} - 2}{-4} = -\frac{e^{2iz} + e^{-2iz}}{4} + \frac{1}{2}$$
$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + e^{-2iz} + 2}{4} = \frac{e^{2iz} + e^{-2iz}}{4} + \frac{1}{2}$$

Therefore, $\sin^2 z + \cos^2 z = 1$ for any $z \in \mathbb{C}$

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2. Suppose $\lim_{z \to z_0} f(z) = 0$ and there exists a positive real number $M$ such that $|g(z)| \leq M$ for all $z$ in some neighborhood of $z_0$. Show that $\lim_{z \to z_0} f(z)g(z) = 0$.

**Solution.** By assumption, there is some $\delta > 0$ such that

$$|g(z)| \leq M \quad \text{whenever } |z - z_0| < \delta.$$

Let $\epsilon > 0$. Since $\lim_{z \to z_0} f(z) = 0$, there is some $\delta_1 > 0$ such that

$$|f(z)| < \frac{\epsilon}{M} \quad \text{whenever } 0 < |z - z_0| < \delta_1.$$

Therefore, if $0 < |z - z_0| < \min\{\delta, \delta_1\}$, then

$$|f(z)g(z)| < \frac{\epsilon}{M} \cdot M = \epsilon.$$

This shows $\lim_{z \to z_0} f(z)g(z) = 0$. ◀

3. Show that the following are entire functions by computing the partial derivatives of their real and imaginary parts and verifying the Cauchy-Riemann equations:

(a) $f(z) = \sin z$.

(b) $f(z) = \cos z$.

(c) $f(z) = \sinh z := \frac{e^z - e^{-z}}{2}$.

(d) $f(z) = \cosh z := \frac{e^z + e^{-z}}{2}$.

**Solution.** Let $z = x + iy$, where $x, y \in \mathbb{R}$

(a)

$$\begin{aligned} f(z) = \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} \\ &= \frac{1}{2i}(e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)) \\ &= \frac{1}{2}(-ie^{-y}\cos x + e^{-y}\sin x + ie^{y}\cos x + e^{y}\sin x) \end{aligned}$$

Therefore, $u(x, y) = \frac{1}{2}(e^y + e^{-y})\sin x$ and $v(x, y) = \frac{1}{2}(e^y - e^{-y})\cos x$. Note that

$$\begin{aligned} u_x &= \frac{1}{2}(e^y + e^{-y})\cos x & \qquad u_y &= \frac{1}{2}(e^y - e^{-y})\sin x \\ v_x &= -\frac{1}{2}(e^y - e^{-y})\sin x & \qquad v_y &= \frac{1}{2}(e^y + e^{-y})\cos x \end{aligned}$$

Clearly, the Cauchy-Riemann equations hold. Since the partial derivatives are continuous, the function $f$ is complex differentiable.

(b) Similar to above, note that $f(z) = \frac{e^{iz}+e^{-iz}}{2}$. The real part $u(x, y)$ and the imaginary part $v(x, y)$ are

$$\begin{aligned} u(x, y) &= \frac{1}{2}(e^y + e^{-y})\cos x \\ v(x, y) &= -\frac{1}{2}(e^y - e^{-y})\sin x \end{aligned}$$

and the partial derivatives are

$$\begin{aligned} u_x &= -\frac{1}{2}(e^y + e^{-y})\sin x & \qquad u_y &= \frac{1}{2}(e^y - e^{-y})\cos x \\ v_x &= -\frac{1}{2}(e^y - e^{-y})\cos x & \qquad v_y &= -\frac{1}{2}(e^y + e^{-y})\sin x \end{aligned}$$

(c) For $f(z) = \sinh z = \frac{e^z - e^{-z}}{2}$, similar to the calculation in part (a), we have

$$\begin{aligned} u(x, y) &= \frac{1}{2}(e^x - e^{-x})\cos y \\ v(x, y) &= \frac{1}{2}(e^x + e^{-x})\sin y \end{aligned}$$

and the partial derivatives are

$$\begin{aligned} u_x &= \frac{1}{2}(e^x + e^{-x})\cos y & \qquad u_y &= -\frac{1}{2}(e^x - e^{-x})\sin y \\ v_x &= \frac{1}{2}(e^x - e^{-x})\sin y & \qquad v_y &= \frac{1}{2}(e^x + e^{-x})\cos y \end{aligned}$$

(d) For $f(z) = \cosh(z) = \frac{e^z + e^{-z}}{2}$, similar to the calculation in part (a), we have

$$u(x,y) = \frac{1}{2}(e^x + e^{-x})\cos y$$
$$v(x,y) = \frac{1}{2}(e^x - e^{-x})\sin y$$

and the partial derivatives are

$$u_x = \frac{1}{2}(e^x - e^{-x})\cos y \qquad u_y = -\frac{1}{2}(e^x + e^{-x})\sin y$$
$$v_x = \frac{1}{2}(e^x + e^{-x})\sin y \qquad v_y = \frac{1}{2}(e^x - e^{-x})\cos y$$

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4. Let $f$ be a function on a domain $D \subset \mathbb{C}$ such that both $f$ and $\bar{f}$ are analytic. Show that $f$ must be a constant function.

**Solution.** Assume $f(z) = u(x,y) + iv(x,y)$, then $\bar{f} = u(x,y) - iv(x,y)$.
By the Cauchy-Riemann equations for the function $f$, we have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

For the function $\bar{f}$, we have

$$u_x = -v_y \quad \text{and} \quad u_y = v_x.$$

These yield $u_x = v_y = 0$ and $u_y = v_x = 0$. Therefore, both $u, v$, and hence the function $f$ are constant functions. ◀