THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 6
Due Date: 30th April 2020

Compulsory Part

1. Use residues to evaluate the following improper integrals:

   (a) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx;$

   (b) $\displaystyle\int_0^\infty \frac{x\sin 2x}{x^2+3}dx.$

2. Use residues to show that

   (a) P.V. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x^2+4x+5}dx = -\frac{\pi}{e}\sin 2;$

   (b) $\displaystyle\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3}$, where $a > 1$.

3. Suppose that f is analytic on and inside a positively oriented simple closed contour γ, and has no zeros on γ. If f has n zeros $z_1, z_2, \ldots, z_n$ inside γ, where z_k is of multiplicity m_k for each k, show that

$$\int_\gamma \frac{zf'(z)}{f(z)}dz = 2\pi i \sum_{k=1}^n m_k z_k.$$

4. Determine the number of zeros, counted with multiplicities, of:

   (a) $z^6 - 6z^4 + 2z^3 - z$ inside $|z| = 1$;

   (b) $z^5 - 3z^3 - z + 1$ inside $|z| = 2$.

5. Prove that $z = 1 - e^{-z}$ has exactly one solution in the right half-plane.

Optional Part

1. Use residues to evaluate the following improper integrals

   (a) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+4}dx;$

   (b) $\displaystyle\int_0^\pi \frac{d\theta}{5+4\sin\theta};$

   (c) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx;$

   (d) P.V. $\displaystyle\int_{-\infty}^\infty \frac{dx}{2x^2+2x+1};$

(e) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin 2x}{2x^2+2x+1}dx;$

(f) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2-1}dx.$

2. Using the fact that $\sin^3 x = \text{Im}\left(\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x} - \frac{1}{2}\right)$, evaluate P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}dx.$

3. Use residues to show that

(a) $\displaystyle\int_0^{\infty} \frac{x^2dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200};$

(b) $\displaystyle\int_0^{\infty} \frac{x^a}{(x^2+1)^2}dx = \frac{(1-a)\pi}{4\cos(a\pi/2)}$, where $-1 < a < 3$.

4. Use Rouché's theorem to show that all the zeros of $z^5 + 3z^2 + 7$ are contained inside the open disk $|z| < 2$.