Chapter S: Centrality in Social Network

S.1 Centralities

One basic but essential measure in social network analysis is centrality. There are 4 common centrality measures which are degree centrality, betweenness centrality, closeness centrality, and eigenvector centrality. They have been employed to understand the roles of certain nodes in networks.

Degree centrality	Average degree of a vertex
Betweenness centrality	Extent to which a particular vertex lies on the shorest path between other vertices
Closeness centrality	The average of the shortest distances to all other vertices
Eigenvector centrality	A measure of the extent of which a vertex is connected to influential other vertices. Related concept is Google's Page Rank

Let $G = (V, E)$ be a simple connected (p, q)-graph. The degree centrality of a vertex is defined to be its degree. But for comparison, some article defines the *degree centrality* of a vertex x by

$$C_D(x) = \frac{\deg(x)}{p-1}.$$

This makes the scale between 0 and 1.

Degree centrality `https://www.youtube.com/watch?v=iiVeQkIELyc`

Suppose $x, u, v \in V$ are distinct vertices in G. Let $g_{u,v}$ denote the number of shortest path between u and v; and $g_{u,v}(x)$ denote the number of shortest path between u and v that pass through x.

The *betweenness centrality* of x is defined by

$$C_B(x) = \frac{1}{\binom{n-1}{2}} \sum_{\substack{u \neq v \\ x \notin \{u,v\}}} \frac{g_{u,v}(x)}{g_{u,v}}.$$

Betweenness centrality `https://www.youtube.com/watch?v=0CCrq62TF7U&t=239s`

Betweenness centrality `https://www.youtube.com/watch?v=ptqt2zr9ZRE&t=1s`

Let $d(u, v)$ denote the distance from u to v, where $u, v \in V$. For $x \in V$, the *closeness centrality* of x is defined by

$$C_C(x) = \frac{p-1}{\sum\limits_{v \in V \setminus \{x\}} d(x, v)}.$$

Closeness centrality `https://www.youtube.com/watch?v=0unzqsPaPk8`

Eigenvector centrality: Score of a page is proportional to the sum of the scores of pages linked to it.

Let A be the adjacent matrix of a connected graph G. Since A is symmetric, A is diagonalizable over $\mathbb{R}$. From Perron-Frobenius Theorem, we know that A is nonnegative definite. Let $\rho(A) = \rho$ be the largest eigenvalue which is called *Perron-Frobenius eigenvalue.* Then ρ is a simple eigenvalue, i.e., the eigenspace corresponding to ρ is of dimension one. Let $\boldsymbol{v}$ be the eigenvector corresponding to ρ. Then all entries of $\boldsymbol{v}$ are positive. So we may assume that $\boldsymbol{v}$ is unit length. Note that, $\boldsymbol{v}$ is the only unit eigenvector having this property.

The entry of $\boldsymbol{v}$ corresponding to the vertex is called the *eigenvector centrality* of that vertex.

Beginner `https://www.youtube.com/watch?v=9vs1zSqd070`

Eigenvector centrality `https://www.youtube.com/watch?v=q8oBWwS2wAQ`

S.2 Page Rank

A concept related to eigenvector centrality is Google's Page Rank.

Following is the Page Rank Algorithm: Given a (p, q)-digraph $\overrightarrow{G} = (V, E)$, let

$$h_{u,v} = \begin{cases} \frac{1}{\deg^+(u)} & \text{if } (u, v) \in E; \\ 0 & \text{otherwise,} \end{cases}$$

for $u, v \in V$.

Let $H = (h_{u,v})^T$ and $\boldsymbol{v} = \frac{1}{p}\mathbf{1}$, where $\mathbf{1}$ is the p-vector whose entries are 1. Let $\boldsymbol{v}_n = H^n\boldsymbol{v}$. Then rank of the vertices follow the natural order of their corresponding entries of $\boldsymbol{v}_n$ for some iteration $n \geq 1$, the largest entry corresponding to the highest rank.

Explanation `https://www.youtube.com/watch?v=MG0fIXfrT9A`

Page Rank Algorithm `https://www.youtube.com/watch?v=P8Kt6Abq_rM`

Original Formula `https://www.youtube.com/watch?v=pA1Q1myuScs`

Matrix Representation `https://www.youtube.com/watch?v=kSmQbVxqOJc`