Chapter 4: Eulerian and Hamiltonian Graphs

4.1 Eulerian Graphs

Definition 4.1.1: Let G be a connected graph. A trail contains all edges of G is called an *Euler trail* and a closed Euler trial is called an *Euler tour* (or *Euler circuit*). A graph is *Eulerian* if it contains an Euler tour.

Lemma 4.1.2: *Suppose all vertices of G are even vertices. Then G can be partitioned into some edge-disjoint cycles and some isolated vertices.*

Theorem 4.1.3: *A connected graph G is Eulerian if and only if each vertex in G is of even degree.*

Corollary 4.1.4: *A connected graph G has an Euler trail if and only if at most two vertices of G have odd degrees.*

Corollary 4.1.5: *For any graph G, the following statements are equivalent:*

1. *All vertices of G are of even degree.*

2. *G is a union of edge-disjoint cycles.*

Fleury's Algorithm
Given an Eulerian graph $G = (V, E)$.

Step 1. Choose any vertex v_0 in G and set $W_0 = v_0$.

Step 2. Suppose the trail $W_i = v_0 e_1 v_1 \cdots e_i v_i$ has been chosen. If there are no edges in $E \setminus \{e_1, \dots, e_i\}$ incident with v_i, then stop. Otherwise, choose an edge e_{i+1} from $E \setminus \{e_1, \dots, e_i\}$ such that

(i) e_{i+1} is incident with v_i and

(ii) unless there is no alternative, e_{i+1} is not a bridge of $G_i = G - \{e_1, \dots, e_i\}$.

Step 3. Define $W_{i+1} = W_i e_{i+1} v_{i+1}$ and go back to Step 2.

It is easy to see that the output of Fleury's algorithm must be a trail.

Theorem 4.1.6: *Fleury's algorithm produces an Euler tour in an Eulerian graph.*

Note that if G contains exactly two odd vertices, then the Fleury's algorithm produces an Euler trail by choosing one of the odd vertices at Step 1. Therefore, we have

Corollary 4.1.7: *If G is a connected graph containing exactly two odd vertices, then a trail produced from the Fleury's algorithm is an Euler trail.*

Chinese Postman Problem: A postman picks up mails at the post office, delivers them and then returns to the post office. He must, of course, traverse all the streets in his area at least once. Subject to this condition, he wants to choose a route that minimize his energy consumption.

This problem can be modeled as a graph theory problem. We construct a weighted graph G where each edge represents a street in the postman's area, each vertex represents a junction of streets and the weight assigned to each edge is the length of the street between junctions. In general, the graph is not

Eulerian. As mentioned before, we need to duplicate some edges to make the graph becomes Eulerian and then apply Fleury's Algorithm to find the tour. Our aim is to find a minimum-weighted tour for this graph and such a tour is called an *optimal tour*.

Given a weighted graph with non-negative weight function W. An edge e is said to be *duplication* when its ends are joined by a new edge with the weight $W(e)$. Hence the Chinese Postman Problem is rephrased as follows.

Given a connected weighted graph with non-negative weight function W,

(1) find, by duplicating edges if necessary, a weighted Eulerian supergraph G^* of G such that the sum of weights of the duplicated edges is as small as possible, that is,

$$\sum_{e\in E(G^*)\setminus E(G)} W(e)$$

is minimum, and

(2) find an Euler tour in G^*.

An appropriate route can be found by first identifying the set S of odd vertices. Then for each pair $u, v \in S$, find the length of a shortest path joining u and v (this can be found by using Dijkstra's algorithm, which will be discussed in later chapter). The next stage is to pair up the vertices in S so that the sum of the distances between those pairs is minimized (this corresponds to find a perfect matching with minimum weight).

In this stage we will only consider a special case: G has exactly two odd vertices.

Theorem 4.1.8: *Suppose G is a weighted connected graph with exactly two odd vertices u and v. Suppose G^* is a supergraph of G such that G^* is Eulerian and the total weight of the duplicated edges is as small as possible. Then the duplicated edges form a shortest (u, v)-path in G.*

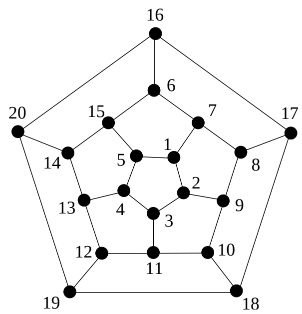
4.2 Hamiltonian Graphs

Definition 4.2.1: A graph with a spanning path is called *traceable* and this path is called a *Hamiltonian path*. A graph with a spanning cycle is called *Hamiltonian* and this cycle is known as a *Hamiltonian cycle*.

It is clear that Hamiltonian graphs are connected; C_n and K_n are Hamiltonian but tree is not Hamiltonian.

A graph G is Hamiltonian if and only if the graph G', where all the loops and multiple edges of G have been removed, is Hamiltonian. Therefore, we can assume all the graphs in this section are simple and connected.

Example 4.2.2: The graph on left hand side of the following figure is the regular dodecahedron which contains 20 vertices, and the graph on right hand side is a plane drawing of it.

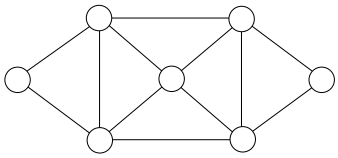


This graph is Hamiltonian since

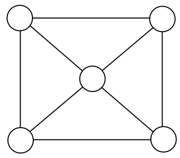
$$1, 2, 3, 4, 5, 15, 14, 13, 12, 11, 10, 9, 8, 17, 18, 19, 20, 16, 6, 7, 1$$

is a Hamiltonian cycle.

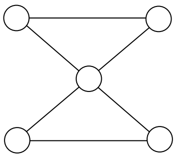
The following graphs show that the concept of Eulerian and Hamiltonian are independent.



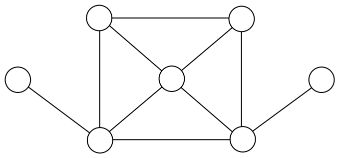
Both Eulerian and Hamiltonian



Hamiltonian but not Eulerian



Eulerian but not Hamiltonian

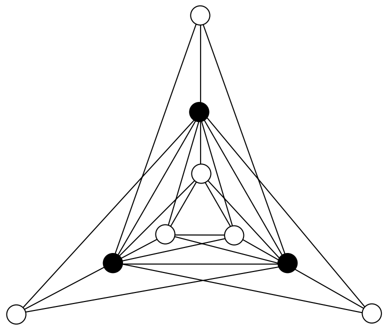


Neither Eulerian nor Hamiltonian

Unlike Eulerian problem, we have not found a necessary and sufficient condition for Hamiltonian problem. Mathematicians only find some sufficient conditions and necessary conditions separately.

Theorem 4.2.3: *If G is a simple Hamiltonian graph, then for each non-empty set $S \subseteq V(G)$, we have $\omega(G - S) \leq |S|$.*

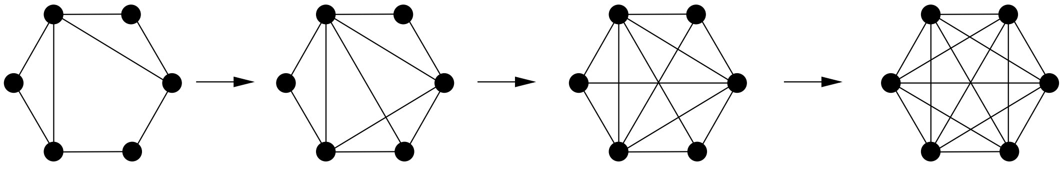
Example 4.2.4: The graph below contains 4 components when the three black vertices are removed. Therefore, by Theorem 4.2.3, this graph is not Hamiltonian.



Bondy and Chvátal* in 1976 modified the proof of Dirac's (Corollary 4.2.8) and obtained the following stronger sufficient condition for Hamiltonian graph.

Theorem 4.2.5 (Bondy and Chvátal, 1976)**:** *Let G be a simple graph on p vertices. Suppose u and v are two nonadjacent vertices such that $\deg(u) + \deg(v) \geq p$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Let G be a simple graph of order p. The *closure* of G is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least p until no such pair remains. We denote the closure of G by $c(G)$. The following is an example.



*J.A. Bondy and V. Chvátal, A method in graph theory, *Discrete Math.*, **15** (1976), 111-136.

Lemma 4.2.6: *The closure $c(G)$ of a graph G is well defined.*

By Theorem 4.2.5 we have the following three corollaries.

Corollary 4.2.7: *A simple graph G is Hamiltonian if and only if $c(G)$ is Hamiltonian.*

Corollary 4.2.8 (Dirac, 1952)**:** *Let G be a simple graph with $p \geq 3$ vertices. If $\deg(u) \geq \frac{p}{2}$ for every $u \in V(G)$, then G is Hamiltonian.*

Corollary 4.2.9: *Let G be a simple graph with $p \geq 3$ vertices. If every pair of nonadjacent vertices u and v has the property $\deg(u) + \deg(v) \geq p$, then G is Hamiltonian.*

In order to verify a graph being Hamiltonian, we have to check whether all pairs of nonadjacent vertices satisfy the condition stated in Theorem 4.2.5. Chinese mathematician Genghua Fan provided a weaker condition in 1984, which only needed to check whether every pairs of vertices of distance 2 satisfy the so-called Fan's condition.

Theorem 4.2.10 (Fan, 1984)**:** *Let G be a 2-connected graph of order p, where $p \geq 3$. If each pair of vertices u and v of distance 2 satisfies $\max\{\deg(u), \deg(v)\} \geq \frac{p}{2}$, then G is Hamiltonian.*

Subsequently, mathematicians have proposed some sufficient conditions for Hamiltonian graph, and even extend to digraph.

Travelling Salesman Problem: A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times (or costs) between towns, how should he plan his trip so that he visits each town exactly once in minimum time (or cost)?

We can represent the salesman's territory by a weighted graph G where the vertices correspond to the towns and two vertices are joined by a weighted edges if and only if there is a directed flight between the corresponding towns. If two towns do not have any directed flight, we may join an edge between the corresponding towns with infinite weight.

So we may assume the weighted graph is complete, which is Hamiltonian. A Hamiltonian cycle of minimum weight is called an *optimal cycle*.

A complete weighted graph G with p vertices has $(p-1)!$ Hamiltonian cycles and half of them are equivalent because of symmetry. Therefore, we need to check $\frac{(p-1)!}{2}$ Hamiltonian cycles if brute-force approach is adopted. Until now, there is no efficient algorithm for finding an optimal cycle in G.

4.3 Eulerian and Hamiltonian Digraphs

Definition 4.3.1: Let $\vec{G}$ be a digraph.

1. A directed trail of $\vec{G}$ that contains exactly one copy of each arc of $\vec{G}$ is called a *directed Eulerian trial.*
2. A closed directed Eulerian trail of $\vec{G}$ is called a *directed Eulerian circuit* or (*directed Euler tour*). A digraph that has a directed Eulerian circuit is called an *Eulerian digraph.*
3. A directed path of $\vec{G}$ that contains all the vertices of $\vec{G}$ is called a *directed Hamiltonian path.*
4. A directed cycle that contains all the vertices of $\vec{G}$ is called a *directed Hamiltonian cycle.* A digraph graph that has a directed Hamiltonian cycle is called a *Hamiltonian digraph.*

Theorem 4.3.2: *Let $\overrightarrow{G}$ be a weakly connected digraph.*

1. *$\overrightarrow{G}$ is an Eulerian digraph if and only if for every $u \in V(\overrightarrow{G})$, we have $\deg^-(u) = \deg^+(u)$.*
2. *$\overrightarrow{G}$ has a directed Eulerian trail if and only if $\overrightarrow{G}$ contains at most two vertices, say u and v, have different indegree and outdegree with $\deg^+(u) - \deg^-(u) = 1$ and $\deg^-(v) - \deg^+(v) = 1$.*

Theorem 4.3.3: *A strongly connected digraph G of order p with $\deg^-(u)+\deg^+(u) \geq p$ for all $u \in V(\overrightarrow{G})$ is Hamiltonian.*

Definition 4.3.4: A digraph $\overrightarrow{G}$ is a *tournament* if its underlying graph G is isomorphic to K_p for some positive integer p.

Theorem 4.3.5: *Let $\overrightarrow{G}$ be a tournament on p vertices. Then*

1. *$\overrightarrow{G}$ has a directed Hamiltonian path.*
2. *If $\overrightarrow{G}$ is strongly connected, then $\overrightarrow{G}$ has a directed cycle of length k, for every $k \in \{3, \ldots, p\}$.*

Definition 4.3.6: Let $\overrightarrow{G}$ be a tournament. A vertex $u^* \in V(\overrightarrow{G})$ is called a *king* if every vertex in $V(\overrightarrow{G})$ is reachable from u^* by a directed path of length at most 2.

Theorem 4.3.7: *Every tournament $\overrightarrow{G} = (V, E)$ has a king. In fact, if $u \in V(\overrightarrow{G})$ is of maximum outdegree, then u is a king of $\overrightarrow{G}$.*