MMAT5320 Computational Mathematics

Midterm

22nd October 2019

- There are a total of 4 questions. The total score is **74** points
- Remember to include your name and ID number.
- Please show all of your detailed steps.
- Answer books and draft papers will be collected back.
- The midterm will start from **18:45** and finish at **20:15**.
- Good luck.

Question 1

(a) [3 pts] Let $A \in \mathbb{C}^{m\times m}$, give the definition of

- the adjoint matrix of A;
- A being hermitian;
- A being unitary.

(b) [4 pts] Let $A \in \mathbb{R}^{m\times m}$ be skew symmetric, i.e., $A^\top = -A$. Show that $(I-A)^\top(I-A) = (I-A)(I-A)^\top$ and use this to prove that $(I-A)^{-1}(I+A)$ is orthogonal.

(c) [6 pts] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$. Given the definition of the induced 2-matrix norm $\|A\|_2$, and prove

$$\|AB\|_2 \le \|A\|_2\|B\|_2.$$

Question 2

(a) [12 pts] Determine a SVD for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) [2 pts] Show that the product of two orthogonal matrices is also an orthogonal matrix.

(c) [6 pts] Two matrices $A, B \in \mathbb{R}^{m\times m}$ are called *orthogonally equivalent* if there is an orthogonal matrix $P \in \mathbb{R}^{m\times m}$ such that $A = PBP^\top$. Show that if A and B are orthogonally equivalent, then they have the same singular values.

Question 3

(a) [7 pts] Let $\{a_1, \ldots, a_n\}$ be a collection of linearly independent vectors in $\mathbb{R}^m$, with $n < m$. Derive from first principles the formula for the orthogonal projector P to the $\text{span}\{a_1, \ldots, a_n\}$.

(b) [6 pts] For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

compute the orthogonal projector P to range(A), and also the image of the point $(1, 0, 1)^\top$ under P.

(c) [5 pts] Let $P \in \mathbb{R}^{m \times m}$ be a projector. Show that $\|P\|_2 \geq 1$.

Question 4

(a) [12 pts] Find a full QR factorisation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(b) [5 pts] Let $A \in \mathbb{R}^{m \times m}$ be an orthogonal and upper triangular matrix with positive values in its main diagonal. Show that A must be the identity matrix. [You may use without proof that the inverse of an upper triangular matrix is again upper triangular]

(c) [6 pts] Let $A \in \mathbb{R}^{m \times m}$ be a matrix of full rank, and suppose there are two QR factorisations of A, i.e., $A = Q_1 R_1 = Q_2 R_2$ such that the entries of main diagonals on R_1 and R_2 are all positive. Show that $Q_1 = Q_2$ and $R_1 = R_2$.

End of questions

Solutions

Question 1

(a) [1 point each] (i) the adjoint of $A = (a_{ij})$ is $A^* := (\overline{a_{ji}})$ where $\overline{a_{ji}}$ is the complex conjugate of a_{ji}. (ii) A is hermitian if $A^* = A$, and (iii) A is unitary if $A^* = A^{-1}$.

(b) Let A be skew symmetric. Then, by computation

$$(I - A)^\top (I - A) = (I + A)(I - A) \underbrace{=}_{[1 \text{ pt}]} I - A^2 \underbrace{=}_{[1 \text{ pt}]} (I - A)(I + A) = (I - A)(I - A)^\top,$$

which also implies [1 pt]

$$(I - A)^{-1}(I - A^2)(I - A)^{-\top} = I.$$

Then, it holds that [1 pt]

$$(I-A)^{-1}(I+A)[(I-A)^{-1}(I+A)]^\top = (I-A)^{-1}(I+A)(I-A)(I-A)^{-\top}$$
$$= (I-A)^{-1}(I-A^2)(I-A)^{-\top} = I.$$

(c) For a matrix $A \in \mathbb{R}^{m\times n}$, [1 pt]

$$\|A\|_2 := \sup_{x\neq 0, x\in\mathbb{R}^n} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{\|x\|_2=1} \|Ax\|_2.$$

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$ be given. Given $x \in \mathbb{R}^q$, $x \neq 0$, define $y = x/\|x\|_2$. Then,

$$\|By\|_2 = \frac{\|Bx\|_2}{\|x\|_2} \quad\Longrightarrow\quad \|B\|_2 \geq \frac{\|Bx\|_2}{\|x\|_2}. \quad \text{[2 pts]}$$

Hence, setting $z = Bx$, we can infer that

$$\|ABx\|_2 = \|Az\|_2 \leq \|A\|_2\|z\|_2 = \|A\|_2\|Bx\|_2 \leq \|A\|_2\|B\|_2\|x\|_2, \quad \text{[2 pts]}$$

and so [1 pt]

$$\|AB\|_2 = \sup_{x\neq 0} \frac{\|ABx\|_2}{\|x\|_2} \leq \|A\|_2\|B\|_2.$$

Question 2

(a) Compute [1 pt]

$$A^\top A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

with characteristic polynomial $(2-x)^2 - 4 = x^2 - 4x = 0$. The eigenvalues are $\sigma_1^2 = 4$ and $\sigma_2^2 = 0$ [2 pts]. The eigenvector w_1 and normalised vector v_1 for σ_1^2 are [2 pts]

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We choose [2 pts; 1 for v_2 and 1 for V]

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad\Longrightarrow\quad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Furthermore, [1 pt]

$$u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and we choose [1 pt]

$$u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad\Longrightarrow\quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

A SVD of A is [3 pts; 1 point each for $U, \Sigma, V^\top$]

$$A = U\Sigma V^\top = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(b) Let A and B be two orthogonal matrices. Then, [2 pts; 1 pt for transpose formula, 1 pt for using orthogonality]

$$(AB)^\top (AB) = B^\top A^\top AB = I, \quad (AB)(AB)^\top = I$$

and so AB is also orthogonal.

(c) Let A and B be orthogonally equivalent. Let μ be a singular value of A. Then, there is an eigenvector x such that $A^\top Ax = \mu x$ [1 pt]. Now since $A = PBP^\top$ for some orthogonal P, we see that

$$\mu x = A^\top Ax \underbrace{=}_{[1\text{ pt}]} PB^\top P^\top PBP^\top x \underbrace{=}_{[1\text{ pt}]} PB^\top BP^\top x.$$

Setting $y = P^\top x$ and multiplying the above equation by $P^\top$ [1 pt] it holds that [1 pt]

$$\mu y = P^\top PB^\top By = B^\top By.$$

Hence μ is also a singular value of B. [1 pt]

Question 3

(a) Let $A \in \mathbb{R}^{m\times n}$ be the matrix whose columns are $\{a_1, \dots, a_n\}$. Then, A is full rank. If $v \in \mathbb{R}^m$ is an arbitrary vector and P is an orthogonal projector to range(A), we have

- $y = Pv \in \text{range}(A)$ and so there is a vector $x \in \mathbb{R}^n$ such that $y = Ax$. [1 pt]
- $v - y = v - Pv$ is orthogonal to range(A) [1 pt] , and so
- $v - y$ inner product with a_i for $1 \le i \le n$ must all be zero. [1 pt] Namely,

$$A^\top(v - y) = A^\top(v - Ax) = 0 \quad \Longrightarrow \quad A^\top Ax = A^\top v. \quad [1pt]$$

- Hence, $x = (A^\top A)^{-1}A^\top v$ [1 pt] since $A^\top A$ is invertible [1 pt] as A has full rank.
- Then, [1 pt]

$$Pv = y = Ax = A(A^\top A)^{-1}A^\top v \quad \Longrightarrow \quad P = A(A^\top A)^{-1}A^\top.$$

(b) Computing [2 pts; 1 for $A^\top A$, 1 for inverse]

$$A^\top A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \quad \Longrightarrow \quad (A^\top A)^{-1} = \frac{1}{6}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

So [full 3 pts for correct formula, otherwise 1 pt for correct multiplication of any two of the matrices]

$$P = A(A^\top A)^{-1}A^\top = \frac{1}{6}\begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{pmatrix}.$$

The image of the point $(1,0,1)^\top$ is $(1,0,1)^\top$. [1 pt]

(c) Let P be a projector, then $P^2 = P$. For any non-zero vector x, we see that

$$\|Px\|_2 \underbrace{=}_{[1\text{ pt}]} \|P^2x\|_2 \underbrace{\le}_{[1\text{ pt}]} \|P\|_2\|Px\|_2 \underbrace{=}_{[1\text{ pt}]} \|P\|_2^2\|x\|_2.$$

Hence, [2 pts]

$$\|P\|_2 \le \|P\|_2^2 \quad \Longrightarrow \quad \|P\|_2 \ge 1.$$

Question 4

(a) Following the Gram–Schmidt orthonormalisation process: Take [2 pts; 1 for r_{11} and 1 for q_1]

$$a_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad r_{11} = \|a_1\|_2 = \sqrt{2}, \quad q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

Set [2 pts; 1 for v_2 and 1 for r_{22}]

$$v_2 = a_2 - (q_1 \cdot a_2)q_1 = \begin{pmatrix}0\\3\\0\end{pmatrix}, \quad r_{22} = \|v_2\|_2 = 3,$$

and normalise [2 pts; 1 for q_2 and 1 for r_{12}]

$$q_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad r_{12} = q_1 \cdot a_2 = 0.$$

Lastly, [1 pt]

$$v_3 = a_3 - (q_1 \cdot a_3)q_1 - (q_2 \cdot a_3)q_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix},$$

and normalising [4 pts; 1 each for q_3, r_{13}, r_{23} and r_{33}]

$$q_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad r_{33} = \|v_3\|_2 = \sqrt{2}, \quad r_{13} = q_1 \cdot a_3 = \sqrt{2}, \quad r_{23} = q_2 \cdot a_3 = 0.$$

Then, the QR factorisation is [1 pt for correct factorisation]

$$A = \begin{pmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0 & \sqrt{2}\\ 0 & 3 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}.$$

(b) Let A be an orthogonal and upper triangular matrix with positive values in its main diagonal. Then, $A^\top$ must be lower triangular [1 pt], and as A^{-1} is upper triangular, by orthogonality relation $A^\top = A^{-1}$ [1 pt], it holds that A must be diagonal [1 pt]. Moreover, since $a_{ii} = a_{ii}^{-1}$ we must have $a_{ii} = 1$ [1 pt; only award if using $a_{ii} > 0$] for all i (since $a_{ii} > 0$), and thus A is the identity matrix. [1 pt]

(c) Let $A = Q_1R_1 = Q_2R_2$. Then [1 pt]

$$Q_2^\top Q_1 = R_2R_1^{-1}.$$

The left hand side is orthogonal [1 pt], while the right hand side is upper triangular [1 pt], and so by part (b), both sides must be the identity matrix [1 pt]. Hence, [2 pts]

$$\begin{aligned} Q_2^\top Q_1 = I \quad &\Longrightarrow \quad Q_1 = Q_2,\\ R_2R_1^{-1} = I \quad &\Longrightarrow \quad R_2 = R_1.\end{aligned}$$