Ch Quaterion (四元數) (Ch17 of the reference)

Def: A quaternion is a "number" of the form

$$\boxed{a+bi+cj+dk}$$

where $a,b,c,d \in \mathbb{R}$.

i,j,k are square roots of -1.

(i.e. $\boxed{i^2=j^2=k^2=-1}$)

in addition: $\boxed{ijk=-1}$

With usual "addition" and "multiplication" laws except the following

$$\begin{cases} ij=-ji=k \\ jk=-kj=i \\ ki=-ik=j \end{cases}$$

Pf: (of $ij=k$):

$$ij=(-ij)(-1)=(-ij)(k^2)$$

$= -(ijk)k = k$ $\quad\square$

egs: (i) $(1+2i+3j+4k) + (2-3i+4j-5k)$

$= (1+2) + (2+(-3))i + (3+4)j + (4+(-5))k$

$= 3 - i + 7j - k.$

(ii) $(2i+j)(j+k)$

$= (2i+j)j + (2i+j)k$

$= 2ij + j^2 + 2ik + jk$

$= 2k - 1 + 2(-j) + i$

$= -1 + i - 2j + 2k.$

Thm Quaternion multiplication has the following properties:

(a) Associativity: $q(rs) = (qr)s$

(b) Distributivity: $q(r+s) = qr + qs$

(c) Inverses: $\forall$ quaterion $q \neq 0$ $\left(0 \overset{\text{def}}{=} 0+0i+0j+0k\right)$
$\exists$ a quaterion r s.t. $qr = 1$.
(denoted $r = q^{-1}$)

Cartersian Form

$$q = t + xi + yj + zk$$

(analogus to $a+bi$ of a complex number)

Scalar part of $q = t + xi + yj + zk$

is defind by $\boxed{Sq = t}$

Vecta part by $\boxed{Vq = xi + yj + zk}$

Note: $Sq \; (\in \mathbb{R})$ is a real number but

Vq is a quaternion.

Conjugate of $q = t + xi + yj + zk$ is defined as

$$\boxed{\begin{aligned} q^* &= Sq - Vq \\ &= t - xi - yj - zk \end{aligned}}$$

Modulus

$$\boxed{|q| \overset{\text{def}}{=} \sqrt{t^2+x^2+y^2+z^2} \overset{\text{Thm}}{=} \sqrt{qq^*}} \quad (\text{Ex!})$$

If $|q|=1$, q is called a <u>unit quaternion</u>.

If $S'q=0$, then q is called a <u>pure quaternion</u>.

> <u>Lemma</u>: Every pure, unit quaternion is a square root of -1.

<u>Pf</u>: Let q be a pure unit quaternion

then $\quad q = xi+yj+zk$ with

$$|q|^2 = x^2+y^2+z^2 = 1$$

Hence $\quad q^2 = (xi+yj+zk)(xi+yj+zk)$

$$= (x i)(x i) + (y j)(x i) + (z k)(x i)$$
$$+ (x i)(y j) + (y j)(y j) + (z k)(y j)$$
$$+ (x i)(z k) + (y j)(z k) + (z k)(z k)$$
$$= x^2 i^2 + \cancel{(xy)(ji)} + \cancel{(zx)(ki)}$$
$$+ \cancel{(xy)(ij)} + y^2 j^2 + \cancel{(zy)(kj)}$$
$$+ \cancel{(xz)(ik)} + \cancel{(yz)(jk)} + z^2 k^2$$
$$= -x^2 - y^2 - z^2 = -1$$

since $ij = -ji$, $ki = -ik$, $kj = -jk$, &

$$i^2 = j^2 = k^2 = -1$$

$\#$

Note: in fact, we've proved that for <u>pure quaternions</u>

$$q^2 = -|q|^2 .$$

Pure Quaternions as vectors in $\mathbb{R}^3$

If $q = x_1 i + y_1 j + z_1 k$

$r = x_2 i + y_2 j + z_2 k$

then $qr = (x_1 i + y_1 j + z_1 k)(x_2 i + y_2 j + z_2 k)$

$$= -x_1 x_2 + y_1 x_2 (ji) + z_1 x_2 (ki)$$
$$+ x_1 y_2 (ij) - y_1 y_2 + z_1 y_2 (kj)$$
$$+ x_1 z_2 (ik) + y_1 z_2 (jk) - z_1 z_2$$

$$= -x_1 x_2 - y_1 x_2 k + z_1 x_2 j$$
$$+ x_1 y_2 k - y_1 y_2 - z y_2 i$$
$$- x_1 z_2 j + y_1 z_2 i - z_1 z_2$$

$$= -(x_1 x_2 + y_1 y_2 + z_1 z_2)$$
$$+ (y_1 z_2 - z_1 y_2) i - (x_1 z_2 - z_1 x_2) j + (x_1 y_2 - y_1 x_2) k$$

$\therefore$ $$\boxed{\begin{aligned} -S(qr) &= x_1x_2 + y_1y_2 + z_1z_2 \\ &= q \cdot r \quad \text{the \underline{dot product} of } q \text{ \& } r \text{ as 3-vectors.} \end{aligned}}$$

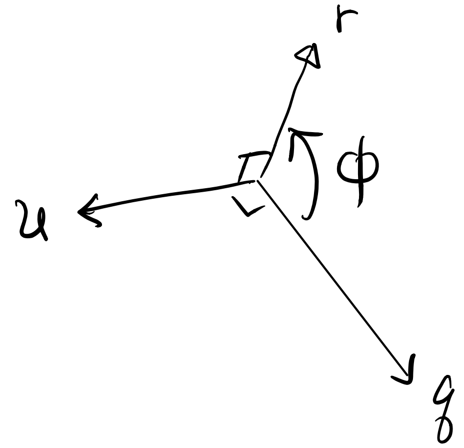
and

$$\boxed{\begin{aligned} V(qr) &= (y_1z_2 - z_1y_2)i - (x_1z_2 - z_1x_2)j + (x_1y_2 - y_1x_2)k \\ &= q \times r \quad \text{the \underline{cross-product} of } q \text{ \& } r \text{ as 3-vectors} \end{aligned}}$$

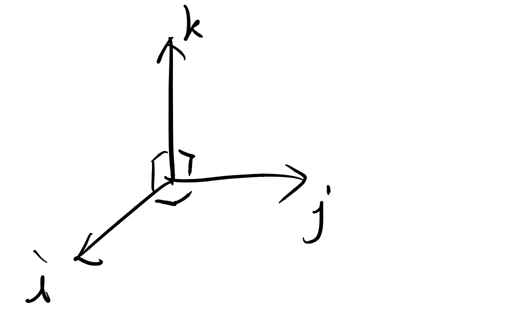
This shows that for pure quaternions q & r

$$\boxed{\begin{aligned} -S(qr) &= q \cdot r = |q||r|\cos\phi \\ V(qr) &= q \times r = \left(|q||r|\sin\phi\right)u \end{aligned}}$$

where
- ϕ = angle between the vectors q & r
- u is a <u>pure unit quaternion</u> representing a unit vector in $\mathbb{R}^3$ perpendicular to q and r, such that (q, r, u) forms a right-handed system



explicit eg: $i \times j = k$



<u>Note</u>: We've used the fact that

modulus of q as pure quaternion

= modulus of q as a 3-vector. (Ex.!)

All together, we have for <u>pure quaternions q & r</u>

$$\boxed{qr = -(q \cdot r) + q \times r}$$

$\uparrow$ quaternion multiplication

$\uparrow$ dot product

$\uparrow$ cross product.

Polar form

Thm: Every quaternion can be represented in the form

$$q = |q|(\cos\theta + u \sin\theta)$$

where $\theta \in \mathbb{R}$; u is a pure unit quaternion ($u^2 = -1$)

Remark:

complex	quaternion
$z = \lvert z\rvert(\cos\theta + i\sin\theta)$	$q = \lvert q\rvert(\cos\theta + u\sin\theta)$
$(\pm i)^2 = -1$ (0-dim'l)	$u^2 = -1$ (2-dim'l)

Pf: Let $q = t + xi + yj + zk$

Set $u = \dfrac{xi + yj + zk}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$

Then u is a pure unit quaternion

and
$$q = t + ru$$
$$= |q|\left(\frac{t}{|q|} + \frac{r}{|q|} u\right)$$

Note $\left(\frac{t}{|q|}\right)^2 + \left(\frac{r}{|q|}\right)^2 = \frac{t^2+(x^2+y^2+z^2)}{|q|^2} = 1$

$\Rightarrow \exists\, \theta \in \mathbb{R}$ s.t. $\frac{t}{|q|} = \cos\theta$ & $\frac{r}{|q|} = \sin\theta$

$\therefore \quad q = |q|(\cos\theta + u\sin\theta)$ ※

Unit Quaternions and Rotations in $\mathbb{R}^3$

Thm (i) Let r be a unit quaternion. Let R be a transformation (of $\mathbb{R}^3$) defined by

$$Rq = rqr^* \qquad \left(\begin{array}{ccc} R: \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; \\ q & \longmapsto & rqr^* \end{array} \right)$$

where q is a pure quaternion.

Then R is a rotation of a 3-dim'l space of pure quaternions about an axis passing thro. the origin.

(ii) Specifically, if the polar form of r is

$$r = \cos\theta + u\sin\theta,$$

where u is a pure unit quaternion.

Then Rq is the pure quaternion obtained by rotating q about u by the angle 2θ.

(iii) Every rotation of 3-dim'l space (about an axis passing thro. the origin) can be expressed in this way.

Pf of (ii) :

Case 1 : $q = u$ (or more generally, $q = \lambda u$, $\lambda \in \mathbb{R}$)

Then $Ru = rur^*$

$= (\cos\theta + u\sin\theta)u(\cos\theta - u\sin\theta)$

$= (u\cos\theta + u^2\sin\theta)(\cos\theta - u\sin\theta)$

$= (u\cos\theta - \sin\theta)(\cos\theta - u\sin\theta)$

$= u\cos^2\theta - \sin\theta\cos\theta - u^2\cos\theta\sin\theta + u\sin^2\theta$

$= u(\cos^2\theta + \sin^2\theta) - \sin\theta\cos\theta + \cos\theta\sin\theta$

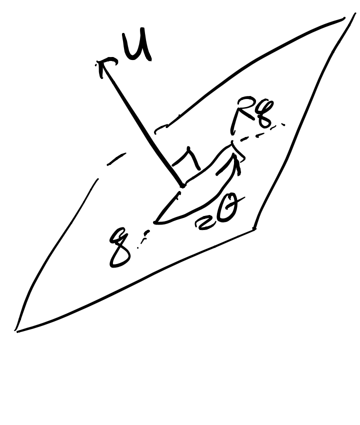
$= u$

$\therefore$ Ru is pure quaternion

and u is fixed point of R.

And immediately, we have $R(\lambda u) = \lambda u$, $\lambda \in \mathbb{R}$

$\therefore$ the axis in the direction of u is fixed by R.



Case 2 q is perpendicular to u $(q \perp u)$

In this case,

$Rq = rqr^*$

$$= (\cos\theta + u\sin\theta)\, q\, (\cos\theta - u\sin\theta)$$

$$= (q\cos\theta + uq\sin\theta)(\cos\theta - u\sin\theta)$$

$$= q\cos^2\theta + uq\sin\theta\cos\theta - qu\cos\theta\sin\theta$$
$$- uqu\sin^2\theta$$

Since u, q are pure quaternions & $q \perp u$,

$$uq = -u\cdot q + u \times q$$
$$= u \times q \qquad (\text{since } u \perp q \Leftrightarrow u\cdot q = 0)$$

$\therefore$ uq is also a pure quaternion.

and also

$$qu = -uq$$

Hence $uqu = u(qu) = u(-uq)$

$$= -u^2 q$$
$$= q.$$

Therefore

$$Rq = q\cos^2\theta + (uq - qu)\cos\theta\sin\theta - uqu\sin^2\theta$$

$$= q\cos^2\theta + (uq + uq)\cos\theta\sin\theta - q\sin^2\theta$$

$$= q(\cos^2\theta - \sin^2\theta) + (2\sin\theta\cos\theta)\, uq$$

$$= (\cos 2\theta)\, q + (\sin 2\theta)\, uq$$

$$\in \mathbb{R}^3 \quad \text{(pure quaternion.)}$$

Also
- $|uq| = |u||q| = |q|$ (Ex!), and
- $(uq)q = (-qu)q$ (by $uq = -qu$)

  $= -q(uq)$

  $\Rightarrow uq \perp q$, and
- $u(uq) = u(-qu) = -(uq)u$

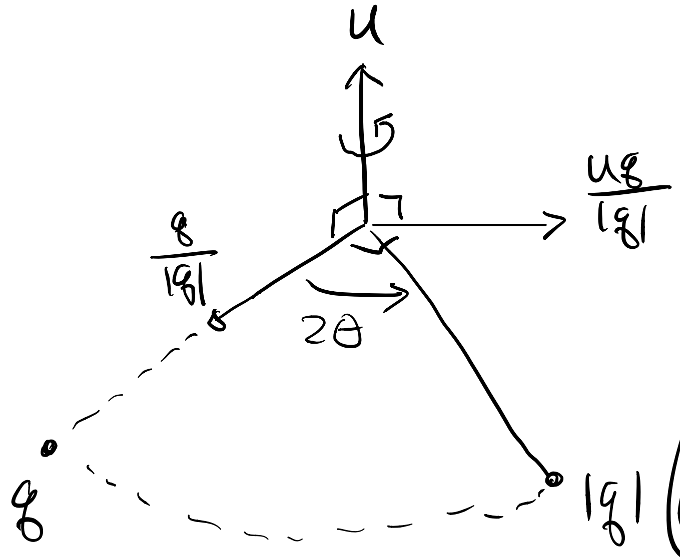
  $\Rightarrow u \perp uq$

Hence $\left\{ \dfrac{q}{|q|}, \dfrac{uq}{|q|} \right\}$ is an orthonormal basis

for the plane perpendicular to u.

$\therefore \; Rq = (\cos 2\theta)\, q + \sin 2\theta \; uq$

is the rotation of q thro. an angle of 2θ about the axis in the direction of u.



$$|q|\left((\cos 2\theta)\frac{q}{|q|} + (\sin 2\theta)\frac{uq}{|q|}\right)$$

$$= (\cos 2\theta)\, q + (\sin 2\theta)\, uq$$

$$= Rq.$$

Case 3 : General pure quaternions

Note that R is a linear transformation

$$\begin{cases} R(q_1+q_2) = Rq_1 + Rq_2, \\ R(\lambda q) = \lambda R q \end{cases} \qquad \left(\begin{array}{c} \forall \text{ pure quaternions } \\ q;\, q_1, q_2 \in \mathbb{R}^3 \\ \lambda \in \mathbb{R} \end{array} \right)$$

Similarly, a rotation in $\mathbb{R}^3$ is also linear.

Denote $\mathcal{O}$ = the rotation thro. an angle of 2θ about the axis of u.

Then any pure quaternion p can be written as

$$p = \lambda u + q$$

where $\lambda \in \mathbb{R}$ and $q \perp u$.

$$\Rightarrow \quad Rp = R(\lambda u + q)$$
$$= \lambda Ru + Rq$$
$$= \lambda \mathcal{O}u + \mathcal{O}q$$
$$= \mathcal{O}(\lambda u + q) = \mathcal{O}p.$$

$\therefore \quad R \equiv \mathcal{O}$.

(Pf of (i) & (iii) are easy from (ii) (Ex!)) ⋕

Remarks:

- $(-r)\, q\, (-r)^* = r q r^*$ (r = unit quaternion)

  Hence $\pm r \longmapsto$ the same rotation in $\mathbb{R}^3$.

- Translation : $Tq = q + b$, where p is pure quaternion.

Final exam up to here!

Corresponding chapters in the textbook are:

Ch 2, 3, 4, 5, 6, 7, 8, 9, 10 & 17

cpx #

Klein geometry

Möbius geometry

hyperbolic geometry

quaternions & rotations in $\mathbb{R}^3$

something in this part.

mainly this part

Ch 18 & 19 3-Dimensional Euclidean and Hyperbolic Geometry (Solid Geometry)

Euclidean Solid Geometry

Def: Let $\mathbb{V} = \{ v = xi + yj + zk : x, y, z \in \mathbb{R} \}$ $(\neq \phi)$

be the set of pure quaternions and

$$\mathbb{IR} = \left\{ \begin{array}{l} T : \mathbb{V} \to \mathbb{V} : Tv = rvr^* + b \\ \text{for some unit quaternion } r \text{ and} \\ \text{pure quaternion } b. \end{array} \right\}$$

be a set of transformations (Euclidean transformations) of $\mathbb{V}$

The pair $(\mathbb{V}, \mathbb{IR})$ models Euclidean Solid Geometry.

Check that this is well-defined. i.e. elements in $\mathbb{IR}$ are really invertible transformations on $\mathbb{V}$, and $\mathbb{IR}$ satisfies the 3 requirments.

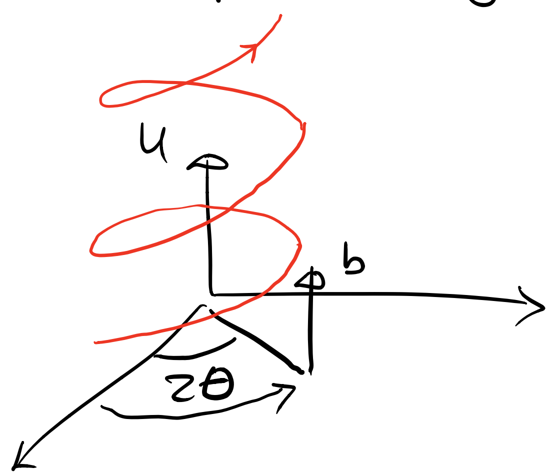
Screw motions

If $r = \cos\theta + u\sin\theta$, u = pure unit quaternion

& b parallel to u

then $Tv = rvr^* + b$

is called a **screw motion**



> **Thm**: Every Euclidean transformation is a screw motion (but centered at different point)

Lemma 1: Every Euclidean transformation with a fixed point is a rotation.

Pf: (i) If O is a fixed point. Then

$$O = TO = rOr^* + b = b$$

$$\Rightarrow b = 0 \ \& \ Tv = rvr^* \text{ is a rotation.}$$

(ii) If q is a fixed point. Let S be a Euclidean transformation such that $Sq = 0$

(for instance : $Sv = v - q$, ie. $r = 1$).

Then STS^{-1} has 0 as fixed point :

$$STS^{-1}(0) = STq = Sq = 0 .$$

$\Rightarrow$ STS^{-1} is a rotation.

by (i) $\Rightarrow$ T is a rotation about an axis passing thro. q.

$$(Tv = r(v-q)r^* + q.)$$ ※

<u>Lemma 2</u> : Let $Tv = rvr^* + b \in \mathbb{R}$

& $r = \cos\theta + u\sin\theta$, $\theta \in \mathbb{R}$

u = unit pure quaternion.

If u and b are perpendicular, then T is a rotation about an axis <u>parallel</u> to u.

<u>Pf</u> : <u>Step 1</u> : $v_0 = \dfrac{1}{2\sin\theta} r^* u b$ is pure quaternion

<u>Pf of step 1</u> : Since u, b pure & $u \perp b$,

we have $ub = -u\cdot b + u \times q = u \times q$

$\therefore$ ub is pure quaternion.

Then $$r^* ub = (\cos\theta + u\sin\theta)^* ub$$
$$= (\cos\theta - u\sin\theta) ub$$
$$= (\cos\theta) ub - u(ub)\sin\theta$$
$$= (\cos\theta) ub + b\sin\theta \quad \text{is pure quaternion}$$

Hence $v_0 = \frac{1}{2\sin\theta} r^* ub$ is also pure quaternion.

<u>Step 2</u>: (i) $bu = -ub$ (proved before)

(ii) $ur = ru$ (note: r <u>not</u> pure)

(iii) $br^* = rb$

<u>Pf of Step 2 (ii)</u> $u(\cos\theta + u\sin\theta) = u\cos\theta - \sin\theta$

$$(\cos\theta + u\sin\theta) u = u\cos\theta + u^2\sin\theta$$
$$= u\cos\theta - \sin\theta.$$

(iii) $br^* = b(\cos\theta + u\sin\theta)^* = b(\cos\theta - u\sin\theta)$
$$= b\cos\theta - bu\sin\theta$$
$$= b\cos\theta + ub\sin\theta \quad \text{(by (i))}$$
$$= (\cos\theta + u\sin\theta) b$$
$$= rb.$$

※

Step 3: v_0 is a fixed point of T

(and hence T is a rotation, by lemma 1)

Pf of Step 3: $T v_0 = r v_0 r^* + b$

$$= r\left(\frac{1}{2\sin\theta} r^* u b\right) r^* + b$$

$$= \frac{1}{2\sin\theta} r r^* u b r^* + b$$

$(|r|^2 = rr^* = 1)$ $$= \frac{1}{2\sin\theta} u b r^* + b$$

(by (iii) of Step 2) $$= \frac{1}{2\sin\theta} u r b + b$$

$$= \frac{1}{2\sin\theta}\left[u(\cos\theta + u\sin\theta) + 2\sin\theta\right] b$$

$$= \frac{1}{2\sin\theta}\left[u\cos\theta - \sin\theta + 2\sin\theta\right] b$$

$$= \frac{1}{2\sin\theta}(u\cos\theta + \sin\theta) b$$

$$= \frac{1}{2\sin\theta}(u\cos\theta - u^2\sin\theta) b$$

$$= \frac{1}{2\sin\theta}(\cos\theta - u\sin\theta) u b$$

$$= \frac{1}{2\sin\theta} r^* u b = v_0$$

#

Final Step: Rotation axis parallel to u.

Pf: Need to show that

$$v_0 + tu \longleftarrow \text{(axis of } u\text{)}$$

are fixed points of T, $\forall t \in (-\infty, \infty)$

To see this:

$$T(v_0+tu) = r(v_0+tu)r^* + b$$
$$= rv_0r^* + t\,rur^* + b$$
$$= (rv_0r^*+b) + t\,rur^*$$
$$= v_0 + t\,urr^* \quad \text{(Step 3 \& (ii) of Step 2)}$$
$$= v_0 + tu \quad (rr^*=1)$$

\#

Proof of the Thm:

Let $Tv = rvr^* + b$, $r = \cos\theta + u\sin\theta$
b = pure quaternion.

Decompose $b = b_1 + b_2$ such that

$$b_1 \perp u, \quad b_2 \parallel u$$

Then $Tv = rvr^* + b$

$$= (rvr^* + b_1) + b_2$$

$\uparrow$ rotation with axis parallel to u (by lemma 2)

$\uparrow$ b_2 translation parallel to u.

Hence T is a screw motion by definition. #

Hyperbolic Solid Geometry

The Half-space Model

Def: Let $\mathbb{U} = \{ q = t + xi + yj : t, x, y \in \mathbb{R}, y > 0 \}$

be the upper half-space.

Let $\mathbb{M}$ be the full Möbius group

$$Tq = (aq+b)(cq+d)^{-1}$$

where a, b, c, d are complex numbers s.t.

$$ad - bc = 1$$

(complex $u + vi$, complex $i \leftrightarrow$ quaternion i)

The pair $(\mathbb{U}, \mathbb{M})$ models <u>3-dim'l hyperbolic geometry</u>.

Note: One needs to show that for $q \in \mathbb{U}$,

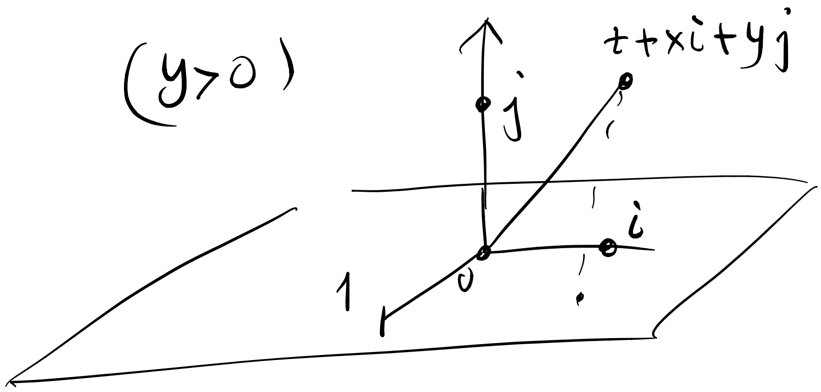
then $Tq \in \mathbb{U}$

(Pf: Omitted, in fact, if $q = z + yj$, $z \in \mathbb{C}$, $y > 0$
then $Tq = (|q|^2 a\bar{c} + b\bar{d} + b\bar{z}\bar{c} + azd) + yj \in \mathbb{U}$)

Comparision:

	hyperbolic plane geometry	hyperbolic solid geometry
points	$x+yi$, $y>0$ upper half plane	$(t+xi)+yj$, $y>0$ $= z+yj$ $(z=t+xi\in\mathbb{C})$ upper half-space
group	Möbius transformation $\leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$ with $a,b,c,d\in\underline{\underline{\mathbb{R}}}$	Möbius transformation $\leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$ with $a,b,c,d\in\underline{\underline{\mathbb{C}}}$

$(y>0)$



Ideals Elements:

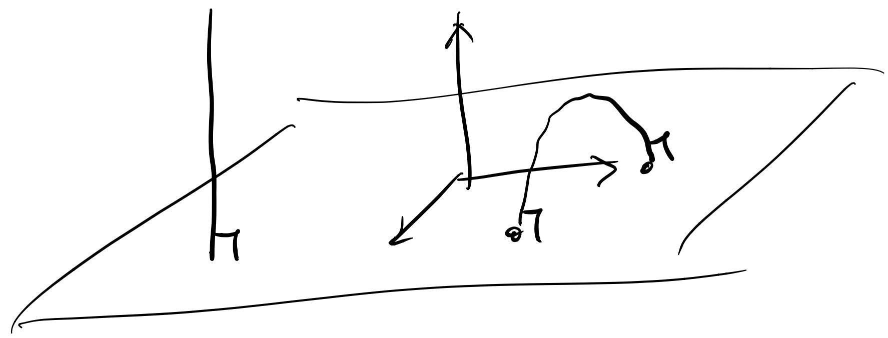
$z=t+xi\in\mathbb{C}$ ideal points (points at infinity)

$(\& \infty\in\hat{\mathbb{C}})$

Planes and Lines

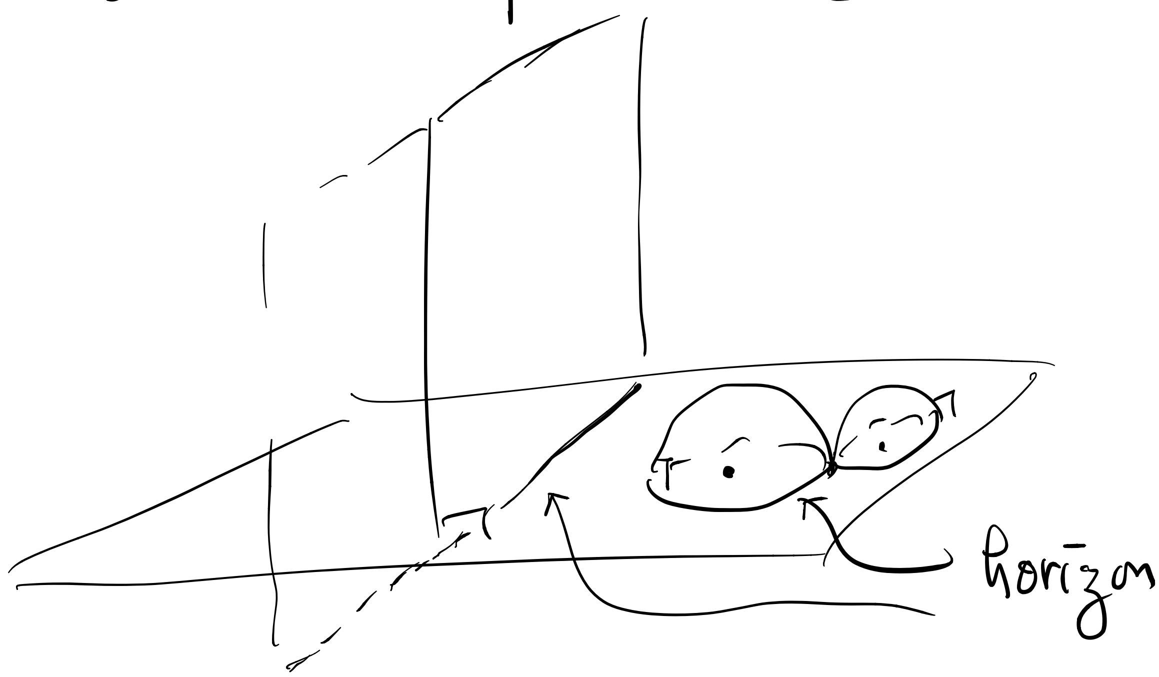
Hyperbolic straight lines

= half circle or Euclidean straight line in $\mathbb{U}$
perpendicular to the "plane at infinity" (tx-plane)



Hyperbolic plane

= Euclidean hemisphere a half-plane
perpendicular to the plane at infinity



The intersection of a hyperbolic plane with the plane at infinity is called the <u>horizon</u> of the plane.

<u>Parallelism</u>

- hyperbolic planes intersect
  $\Rightarrow$ intersection = hyperbolic line
- hyperbolic planes <u>do not</u> intersect
  (i) <u>parallel</u>: horizons are tangent
  (ii) <u>hyperparallel</u>: otherwise

<u>Cycles and Spheres</u>

<u>Cycle</u> = Euclidean circle a straight line in $\mathbb{U}$ that is <u>not</u> perpendicular to the plane at infinity

(hyperbolic circles, horocycles, and hypercycles as in 2-dim.)

Similarly, <u>sphere</u>, <u>horosphere</u> & <u>hyperspheres</u>.
= Euclidean spheres and planes that are not perpendicular to the plane at infinity.

<u>Arc-length</u>: $\gamma : g(s) = t(s) + x(s)i + y(s)j$
(s = parameter, $a \le s \le b$)

$$\boxed{L(\gamma) = \int_a^b \frac{\sqrt{(t'(s))^2 + (x'(s))^2 + (y'(s))^2}}{y(s)}\, dt}$$

$$\boxed{\text{Volume of a solid } R = \iiint_R \frac{dt\,dx\,dy}{y^3}}$$

#