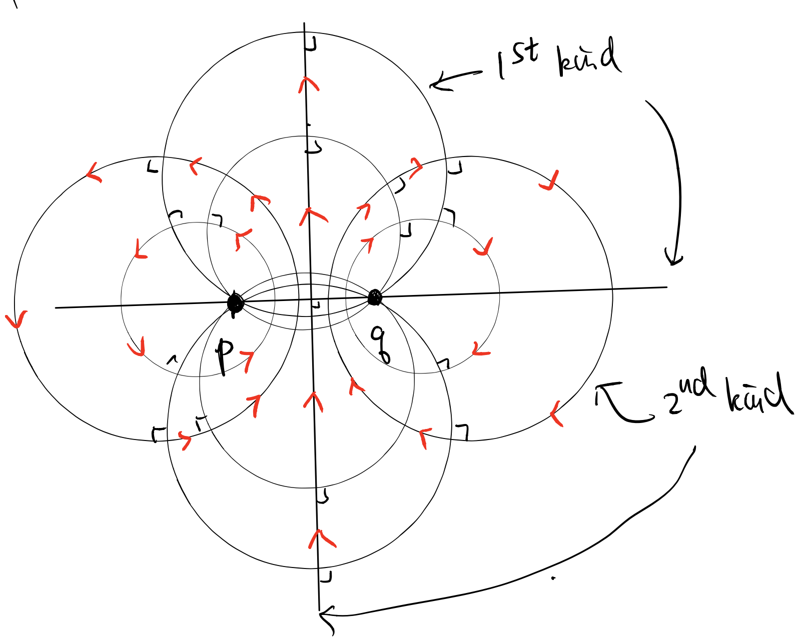
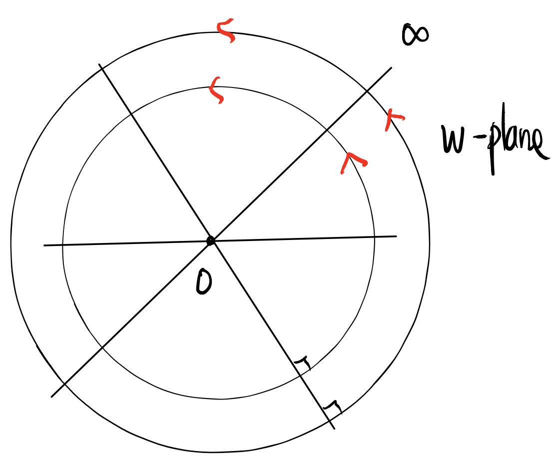
Case 1 Elliptic transformation ($|\lambda| = 1$)

$\lambda = e^{i\theta} \Rightarrow Rw = e^{i\theta} w$ is a rotation about the origin

$\Rightarrow$ the action of T is to move points along the Steiner circles of 2^{nd} kind "around the fixed points".

(Moreover, T sends Steiner circles of 1^{st} kind to (another) Steiner circles of 1^{st} kind.)



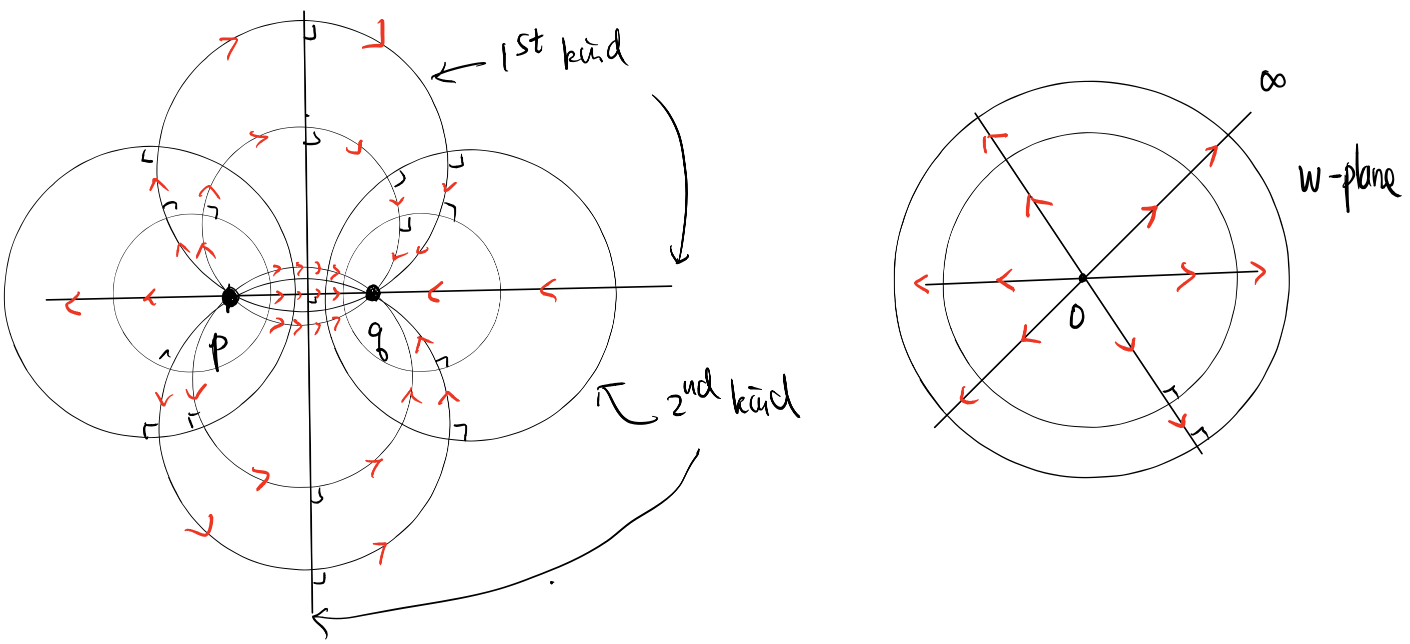


Case 2 Hyperbolic transformation ($\lambda > 0$, $\lambda \in \mathbb{R}$)

$Rw = \lambda w$, $\lambda > 0$ is a homothetic transformation

$\Rightarrow$ the action of T is to move points along the Steiner circle of the 1st kind

(Moreover, T sends Steiner circles of the 2nd kind to (another) Steiner circles of the 2nd kind.)



Case 3 Loxodromic Transformation

$\lambda = ke^{i\theta}$, $k \neq 1$, $k > 0$, and $\theta \neq 0 \pmod{2\pi}$

action of T = a combination of the motions of an elliptic and a hyperbolic transformation.

Conclusions

(1) 2 kinds of Steiner circles
→ generalized polar coordinates for Möbius Geometry.

(2) Möbius transformations with 2 fixed points transform each Steiner circle wrt the fixed points to (itself or another) Steiner circle (of the same kind) wrt the same fixed point.

(3) Simplest types of transformations with 2 fixed points:

(a) Elliptic (rotation): move points along Steiner circles of 2nd kind.

(b) Hyperbolic (scaling): move points along Steiner circles of 1st kind.

(4) Loxodromic = combination of elliptic & hyperbolic

(5) Normal form = expression of the relationship between the transformation and the Steiner circle coordinate system determined by its fixed points.

Parabolic Transformation (Transformation with 1 fixed point)

Let T be a transformation with one fixed point p.

Consider $\quad w = Sz = \dfrac{1}{z-p}$.

Then $S(p) = \infty$

And $R = STS^{-1}$

satisfies

$$\begin{tikzcd} w \in \hat{\mathbb{C}} \arrow[r, "R"] \arrow[d, "S^{-1}"'] \arrow[dr, phantom, "\circlearrowright"] & \hat{\mathbb{C}} \arrow[d, "S^{-1}"] \\ z \in \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd}$$

$$\begin{aligned} R(\infty) &= STS^{-1}(\infty) \\ &= ST(p) = Sp = \infty \end{aligned}$$

Using the form $\; Rw = \dfrac{aw+b}{cw+d}$, $\; a,b,c,d \in \mathbb{C}$ with $ad-bc \neq 0$,

we have

$$R(\infty) = \infty \Rightarrow c = 0 \quad (\Rightarrow d \neq 0, a \neq 0)$$

Hence

$$Rw = \left(\frac{a}{d}\right) w + \left(\frac{b}{d}\right)$$

Since R has no other fixed point (otherwise T will have 2 fixed points.)

$$\Rightarrow \quad w = \left(\frac{a}{d}\right) w + \frac{b}{d} \text{ has no solution in } \mathbb{C}$$

$\Rightarrow \quad \frac{a}{d} = 1$ and $\frac{b}{d} \neq 0$

$\therefore \quad Rw = w + \beta$ for some $\underset{\neq 0}{\beta} \in \mathbb{C}$, which is a translation.

Hence

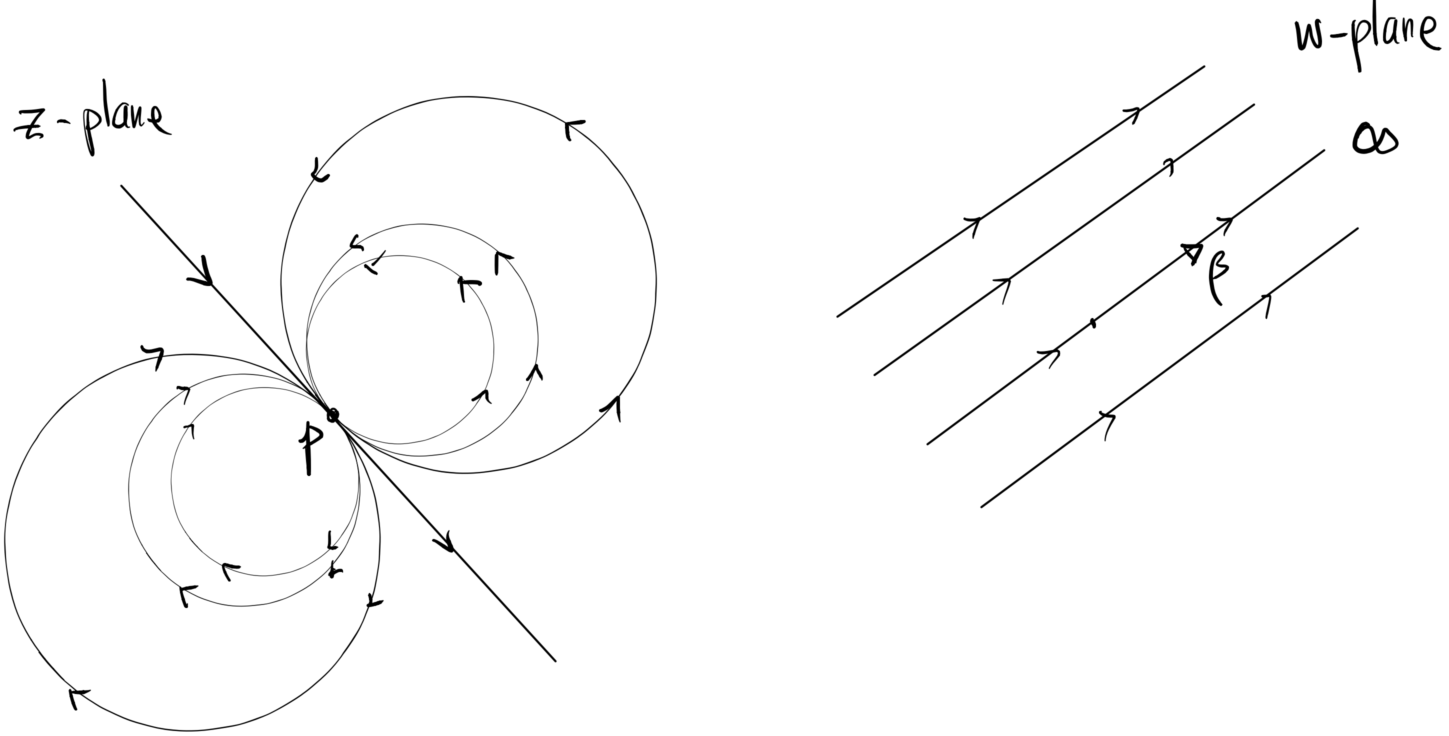
$$STS^{-1}w = Rw = w + \beta$$

$$\Rightarrow \quad STz = Sz + \beta$$

$$\Rightarrow \quad \boxed{\frac{1}{Tz - p} = \frac{1}{z - p} + \beta} \quad (\beta \neq 0)$$

is the <u>normal form</u> of a parabolic transformation

<u>Note</u>: $Rw = w + \beta$ moves point along straight lines parallel to the vector β.



Adding the family of lines orthogonal to line parallel to β, we have a coordinate system on w-plane which gives a coordinate system on the z-plane called a generalized Cartesian coordinate system.