Invariants of Möbius Geometry

- Angle measurement

Möbius transformations are conformal

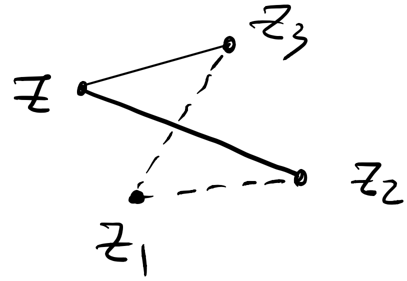
$\Rightarrow$ (Euclidean) angle measure is an invariant of Möbius Geometry.

- Cross Ratio

Def: The cross ratio is the following function of 4 (extended) complex variables:

$$(z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

Remarks (1)



$$\frac{z-z_2}{z-z_3} \Big/ \frac{z_1-z_2}{z_1-z_3}$$

(2) If z_1, z_2, z_3 are held constants, then as a function of z, $Tz = (z, z_1, z_2, z_3)$ is the unique Möbius transformation sending z_1 to 1, z_2 to 0, and z_3 to ∞.

Thm: Let z, z_1, z_2, z_3 be 4 distinct points on $\hat{\mathbb{C}}$, then $\forall\ S \in \mathbb{M}$,

$$(Sz, Sz_1, Sz_2, Sz_3) = (z, z_1, z_2, z_3)$$

Pf: By the remark (2) above,

$Tz = (z, z_1, z_2, z_3)$ is the unique Möbius transformation such that

$$Tz_1 = 1,\ Tz_2 = 0,\ Tz_3 = \infty.$$

Consider the composition $T \circ S^{-1} \in \mathbb{M}$

Note that
$$\begin{cases} T\circ S^{-1}(Sz_1) = Tz_1 = 1 \\ T\circ S^{-1}(Sz_2) = Tz_2 = 0 \\ T\circ S^{-1}(Sz_3) = Tz_3 = \infty \end{cases}$$

$$\Rightarrow \quad T\circ S^{-1}(z) = (z, Sz_1, Sz_2, Sz_3) \quad (\forall z)$$

Therefore

$$Tz = T\circ S^{-1}(Sz) = (Sz, Sz_1, Sz_2, Sz_3)$$
$$\parallel$$
$$(z, z_1, z_2, z_3)$$

$\divideontimes$

Thm: The cross ratio (z, z_1, z_2, z_3) is real if and only if the 4 points lie on a Euclidean circle or straight line (including ∞).

Pf: $(z, z_1, z_2, z_3) \in \mathbb{R}$

$\Leftrightarrow (Tz, Tz_1, Tz_2, Tz_3) \in \mathbb{R}, \ \forall T \in \mathbb{M}$.

Let $T \in \mathbb{M}$ be the Möbius transformation such that $Tz_1 = 1$, $Tz_2 = 0$, $Tz_3 = -1$.

Then

$$\mathbb{R} \ni (z, z_1, z_2, z_3) = (Tz, 1, 0, -1)$$

$$= \frac{Tz - 0}{Tz - (-1)} \cdot \frac{1 - (-1)}{1 - 0}$$

$$= \frac{2Tz}{1 + Tz}$$

If $(z, z_1, z_2, z_3) = 2$, then $Tz = \infty$

If $(z, z_1, z_2, z_3) \neq 2$, then $Tz = \dfrac{(z, z_1, z_2, z_3)}{2 - (z, z_1, z_2, z_3)} \in \mathbb{R}$

In any case, Tz, Tz_1, Tz_2, Tz_3 lie on the x-axis

Therefore, z, z_1, z_2, z_3 lie on a Euclidean circle or a straight line (Since möbius transforms maps lines/circles to lines/circles.)

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Clines

Def: A subset C of the complex plane is a cline if C is a Euclidean circle or Euclidean straight line.

Thm: If C is a cline, then $T(C)$ is a cline, $\forall T \in M$.

(Pf: Ex!)

Remark: All circles and straight lines are congruent to each other in Möbius geometry: (i) circle determined by 3 points

(ii) straight line is just a "circle" passing throught ∞.

(Ex!)

Symmetry

Def: Let C be a cline passing through 3 distinct points $z_1, z_2, \& z_3$. Two points z and z^* are called <u>symmetric with respect to</u> C if

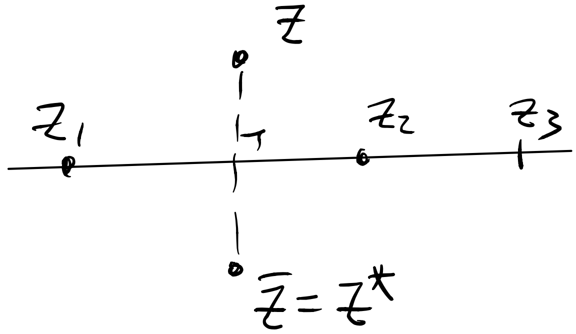
cpx conjugate

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

eg.: If z_1, z_2, z_3 are 3 distinct points on x-axis,

then $\quad z^* = \bar{z}$

which is the usual mirror symmetry of z across the x-axis



(HW1, Q5(a))

<u>Remarks</u>: (i) In this case, we see that one can take any 3 points on the x-axis to give the symmetry wrt x-axis. Similarly, this is true for any cline C.

(ii) z, z^* symmetric wrt C

$\Leftrightarrow$ Tz, Tz^* symmetric wrt $T(C)$ (Ex!)

ie. $$\boxed{T(\underbrace{z^*}_{\text{wrt } C}) = (Tz)^{\overset{*}{\text{wrt } T(C)}}}$$

eg: <u>(Formulae for symmetric points)</u>
(wrt a Euclidean circle)

If $C = \{z : |z-a|^2 = R^2\}$, and $z_1, z_2, z_3 \in C$

Then z, z^* symmetric wrt C

$$\Longleftrightarrow \quad (z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

(Cross-ratio is invariant under Möbius transformations)

$$= \overline{(z-a, z_1-a, z_2-a, z_3-a)}$$

$$= \left(\overline{z-a}, \overline{z_1-a}, \overline{z_2-a}, \overline{z_3-a}\right)$$

($z_1, z_2, z_3 \in C$)

$$= \left(\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right)$$

$$= \left(\frac{\bar{z}-\bar{a}}{R^2}, \frac{1}{z_1-a}, \frac{1}{z_2-a}, \frac{1}{z_3-a}\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}+a, z_1, z_2, z_3\right)$$

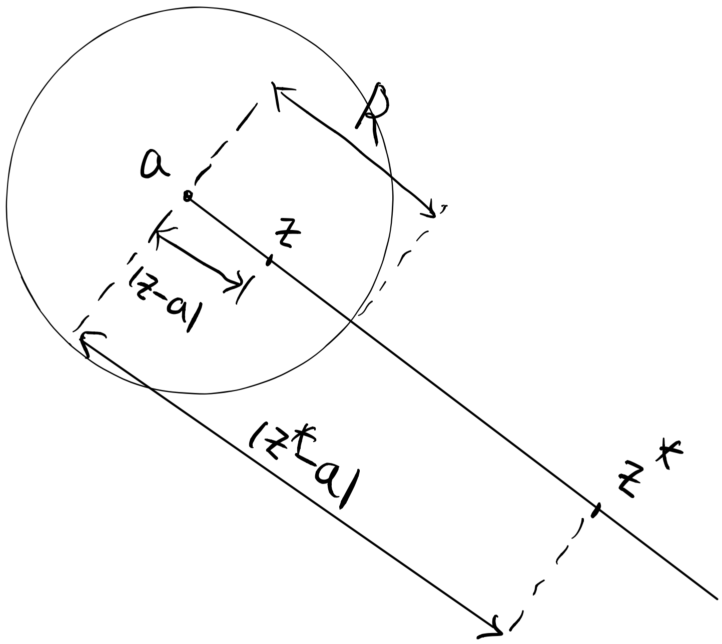
$$\Longleftrightarrow \quad \boxed{z^* = \frac{R^2}{\bar{z}-\bar{a}} + a}$$

or

$$\boxed{z^* - a = \frac{R^2}{|z-a|^2}(z-a)}$$

which implies $\boxed{|z^* - a||z - a| = R^2}$

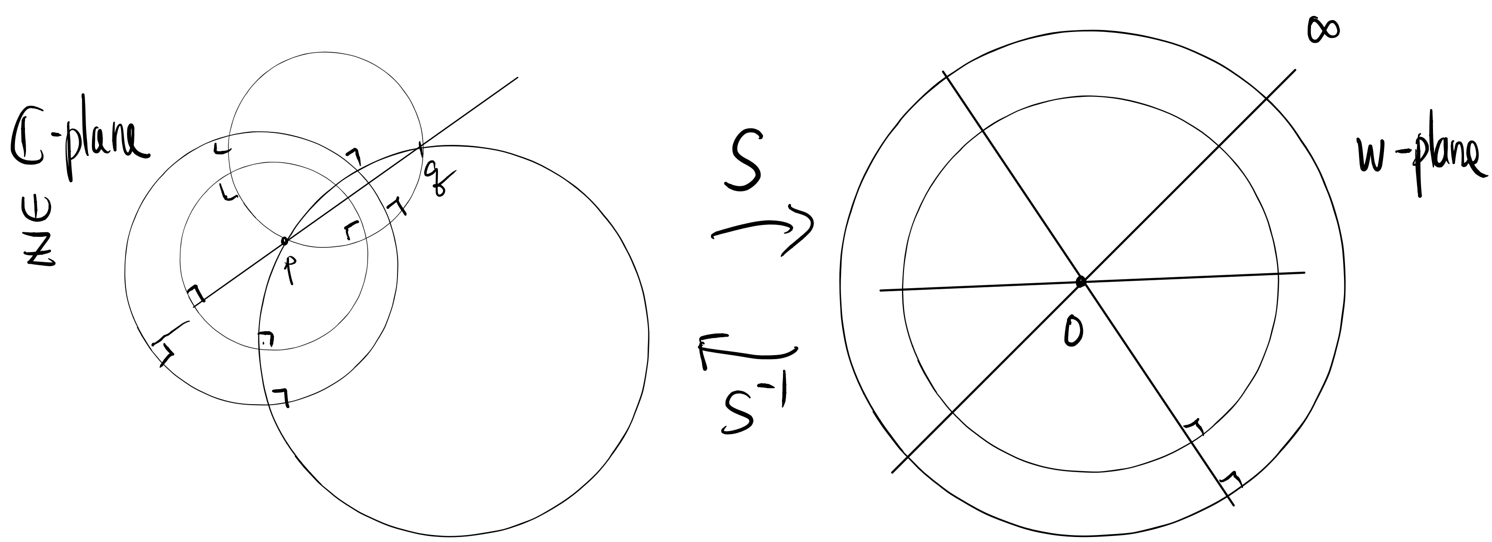
and $\boxed{(z^*)^* = z}$ (Ex!)



Ch6 Steiner Circles

Families of clines

Let $p \neq q \in \mathbb{C}$, the family of all clines passing through p and q is called the Steiner circles of the first kind with respect to points p and q



Consider the transformation

$$w = Sz = \frac{z-p}{z-q}$$

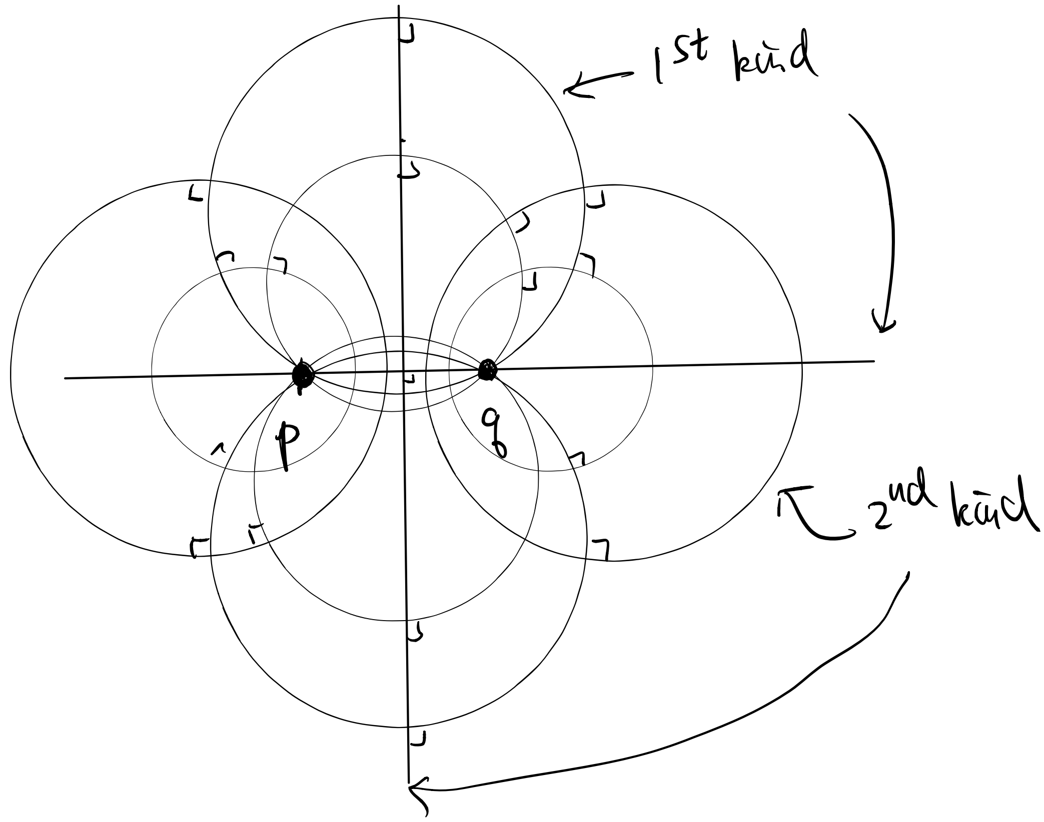
Then
$$\begin{cases} p \overset{S}{\longmapsto} 0 & (\text{ie. } Sp = 0) \\ q \overset{S}{\longmapsto} \infty & (\text{ie. } Sq = \infty) \end{cases}$$

Recall that Möbius transformations takes clines to clines, the image of the circles (clines) in the Steiner circles of the 1^{st} kind wrt p & q form the "Steiner circles" of the 1^{st} kind wrt 0 & ∞.

In the w-plane, it is easy to see that there is another family of clines orthogonal to the "Steiner circles" of the 1^{st} kind: namely the family of circles centered at $w=0$.

The pull back of these circles $\{|w|=k\}$ in the w-plane by S^{-1} form a family of clines on z-plane which is called the Steiner circles of the second kind (wrt p & q)

(also called circles of Apollonius)



By definition, the Steiner circles of 2nd kind (wrt p & q) have the equation:

$$\boxed{\frac{|z-p|}{|z-q|} = k}$$

<u>Remark</u>: The families of Steiner circles of 1st & 2nd kinds can be regarded as a generalization of <u>polar coordinates to Möbius geometry</u>.

Normal Form of a Möbius Transformation

Let T = Möb transformation with 2 fixed points p & q.

Since (a) T fixes p & q and

(b) T maps clines to clines,

T maps clines passing thro. p & q to another clines passing thro. p & q.

$\therefore$ Steiner circles of 1^{st} kind wrt p & q are invariant under T.

$\Rightarrow$ One can show that Steiner circles of 2^{nd} kind are also invariant under T (Ex!)

To see the action of T, we consider again

$$w = Sz = \frac{z-p}{z-q}$$

$$\begin{tikzcd} w \in \hat{\mathbb{C}} \arrow[r, "R"] \arrow[d, "S^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "S^{-1}"] \\ z \in \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd}
\qquad
\left(\begin{tikzcd} z \in \hat{\mathbb{C}} \arrow[r, "T"] \arrow[d, "S"'] & \hat{\mathbb{C}} \arrow[d, "S"] \\ w \in \hat{\mathbb{C}} \arrow[r] & \hat{\mathbb{C}} \end{tikzcd} \right)$$

Let $R(w) = STS^{-1}(w)$ be the lift of T to the (extended) w-plane via S^{-1}.

Then
$$\begin{cases} R(0) = STS^{-1}(0) = ST(p) = S(p) = 0 \\ R(\infty) = STS^{-1}(\infty) = ST(q) = S(q) = \infty \end{cases}.$$

One the other hand, R is of the form

$$Rw = \frac{aw+b}{cw+d}, \text{ for some } a,b,c,d \in \mathbb{C} \text{ with } ad-bc \neq 0.$$

Then
$$\begin{cases} R(0) = 0 \Rightarrow & b = 0 \\ R(\infty) = \infty \Rightarrow & c = 0 \end{cases}$$

$$\therefore \quad Rw = \left(\frac{a}{d}\right) w \qquad \left(a, d \neq 0 \text{ since } 0 \neq ad - bc = ad \right)$$

Write $\lambda = \frac{a}{d} \neq 0$, we have

$$\boxed{Rw = \lambda w}$$

Substituting into $Rw = STS^{-1}w$,

$\Rightarrow$ $$\lambda w = STS^{-1}w$$

$\Rightarrow$ $$\lambda(Sz) = STS^{-1}(Sz) = S(Tz)$$

$$\boxed{\lambda \frac{z-p}{z-q} = \frac{Tz-p}{Tz-q}} \quad (\lambda \neq 0)$$

is called the <u>normal form</u> of T.

We see that T (with two distinct fixed points) can be understood as composition of 3 operations:

(i) sending the fixed points to 0 and ∞.

(ii) multiplication by a nonzero complex constant $\lambda \neq 0$.

(iii) sending 0 and ∞ back to the fixed points.