Abstract Geometries and their Models

<u>Def</u>: Two geometries (S_1, G_1) & (S_2, G_2) are <u>models</u> of the same <u>abstract geometry</u> if there is an invertible covering transformation $\mu : S_1 \to S_2$ such that

$$\boxed{\begin{array}{l} T_1 \in G_1 \Rightarrow \mu \circ T_1 \circ \mu^{-1} \in G_2 \\ T_2 \in G_2 \Rightarrow \mu^{-1} \circ T_2 \circ \mu \in G_1 \end{array}}$$

$$\begin{tikzcd} S_1 \arrow[r, "T_1 \in G_1"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowright"] & S_1 \arrow[d, "\mu"] \\ S_2 \arrow[r, dashed, "\mu \circ T_1 \circ \mu^{-1} \in G_2"'] & S_2 \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, dashed, "\mu^{-1} \circ T_2 \circ \mu \in G_1"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowright"] & S_1 \arrow[d, "\mu"] \\ S_2 \arrow[r, "T_2 \in G_2"'] & S_2 \end{tikzcd}$$

<u>Note</u>: In this case, (S_1, G_1) and (S_2, G_2) are called <u>isomorphic</u> and μ is called an <u>isomorphism</u>.

eg: $\begin{cases} S_1 = \{ z \in \mathbb{C} : |z| < 1 \} \\ G_1 = \{ \text{rotations around the origin} \} \end{cases}$

$\begin{cases} S_2 = \{ z \in \mathbb{C} : |z-5| < 3 \} \\ G_2 = \{ \text{rotations around } z=5 \} \end{cases}$

Then (S_1, G_1) and (S_2, G_2) are geometries. (check!)

Consider $\mu : S_1 \to S_2$, $z \mapsto 3z+5$ (check: $|\mu(z) - 5| < 3$)

It is easy to see $\mu^{-1} : S_2 \to S_1$, $z \mapsto \frac{z-5}{3}$ exists. (check!)

Let $T_1 \in G_1$, i.e. T_1 = rotation around 0

$\Rightarrow T_1 z = e^{i\theta_1} z$, for some $\theta_1 \in \mathbb{R}$.

Then $\forall \, \xi \in S_2$

$$\begin{aligned} \mu \circ T_1 \circ \mu^{-1}(\xi) &= \mu \circ T_1 (\mu^{-1}(\xi)) \\ &= \mu \circ T_1 \left(\frac{\xi - 5}{3} \right) \\ &= \mu \left(e^{i\theta_1} \left(\frac{\xi-5}{3} \right) \right) \end{aligned}$$

$$= 3 \cdot \left(e^{i\theta_1} \left(\frac{z-5}{3} \right) \right) + 5$$

$$= e^{i\theta_1}(z-5) + 5$$

which is a rotation of θ_1 degree around $z=5$

$\therefore \mu \circ T_1 \circ \mu^{-1} \in G_2$

Similarly for the other direction.

$\therefore (S_1, G_1)$ & (S_2, G_2) are isomorphic. #

Ch5 Möbius Geometry

Stereographic

Def: Let $\hat{\mathbb{C}}$ (or $\mathbb{C}^+$) $= \mathbb{C} \cup \{\infty\} \cong S^2$ be the extended complex plane (including ∞), and let $\mathbb{M}$ be the set of transformations of the form

$$w = Tz = \frac{az+b}{cz+d}$$

where $a, b, c, \& \ d \in \mathbb{C}$, and the determinant of T, $ad-bc \neq 0$.

Such a transformation is called a Möbius transformation (or linear fractional transformation).

The pair $(\hat{\mathbb{C}}, \mathbb{M})$ models Möbius Geometry.

Remark: Möbius transformations include all rotations, translations, homothetic transformation, and inversion:

- rotation: $w = e^{i\theta} z$; $a = e^{i\theta}$, $b = c = 0$, $d = 1$
  (then $ad - bc = e^{i\theta} \neq 0$)

- translation : ? (Ex!)

- homothetic transformation : ?

- inversion : $w = \frac{1}{z}$; $a = d = 0,\ b = c = 1$
  (then $ad - bc = -1 \neq 0$)

Conversely

$$w = Tz = \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} - \frac{ad-bc}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right) & , \text{if } c \neq 0 \\ \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right) & , \text{if } c = 0 \end{cases}$$

(check!)

$c = 0 \Rightarrow T =$ composition of a rotation, homothetic transformation and a translation (in general)

$c \neq 0 \Rightarrow T =$ composition of a translation, followed by an inversion, followed by a homothetic transformation and a rotation, and followed by another translation (in general)

Proof of $(\hat{\mathbb{C}}, \mathbb{M})$ is a geometry

(i) $\forall$ Möbius transformation $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$T(\infty) = \begin{cases} \frac{a}{c}, & \text{if } c \neq 0 \\ \infty, & \text{if } c = 0 \end{cases}$$

$$T\left(-\frac{d}{c}\right) = \infty \quad \text{if } c \neq 0$$

$\therefore$ T is well-defined on $\hat{\mathbb{C}}$

(ii) $Id_{\hat{\mathbb{C}}}: z \mapsto z$ (ie. $w = z$; $a = 1 = d$, $b = c = 0$, $ad - bc = 1 \neq 0$)

is a Möbius transformation.

(iii) Let $Tz = \frac{az+b}{cz+d}$, $Sz = \frac{ez+f}{gz+h}$

with $ad - bc \neq 0$ and $eh - gf \neq 0$

then $T \circ S(z) = T(Sz) = \frac{a(Sz)+b}{c(Sz)+d}$

$$= \frac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d}$$

$$= \frac{a(ez+f)+b(gz+h)}{c(ez+f)+d(gz+h)}$$

$$= \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}$$

Note $(ae+bg)(cf+dh)-(af+bh)(ce+dg)$

$= (ad-bc)(eh-gf) \neq 0$

$\Rightarrow$ $T \circ S$ is a Möbius transformation

(iv) From (iii), we see that we can associate a 2×2 complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the Möbius transformation $w = Tz = \frac{az+b}{cz+d}$

Then for $T \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$S \leftrightarrow \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

we have $T \circ S \leftrightarrow \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$\left(\begin{array}{l} \text{and determinant of } T \Longleftrightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \& \qquad \det(T \circ S) = \det(T)\det(S) \end{array} \right)$$

Hence T^{-1} should correspond to

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

$$\therefore \; z = T^{-1}w = \frac{dw-b}{-cw+a} \quad \text{is a Möbius transformation.}$$

All together, $(\hat{\mathbb{C}}, \mathbb{M})$ is a geometry. #

<u>Remark</u>: $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \qquad k \in \mathbb{C}\setminus\{0\}$

$$\Longleftrightarrow \; Tz = \frac{(ka)z + (kb)}{(kc)z + (kd)}$$

$$= \frac{az+b}{cz+d} \Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Two different matixes associated to the same transfamation! (In fact, infinitely many)

To overcome this, we can normalize so that all Möbius transformation $w = Tz = \dfrac{az+b}{cz+d}$ satisfying $\boxed{ad-bc=1}$

But still, we have

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow Tz = \frac{az+b}{cz+d}$$

since $\det\left[-\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

Then the group of Möbius transformations corresponds to the matrix group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1, \ a, b, c, d \in \mathbb{C} \right\}$

up to ± 1 (a multiplication of $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

The group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1; a, b, c, d \in \mathbb{C} \right\}$

is called the special linear group with complex entries $SL(2, \mathbb{C})$.

$$\Rightarrow \quad \mathbb{M} = SL(2,\mathbb{C}) \big/ \{\pm I\} \quad \text{(a quotient group)}$$

$$= PSL(2,\mathbb{C}) \quad \text{(projectivized special linear group)}$$

Fixed Points of Möbius Transformations

Def. A fixed point of a (Möbius) transformation T is a point z $(\in \hat{\mathbb{C}})$ such that

$$Tz = z.$$

eg: Let $Tz = \dfrac{az+b}{cz+d}$ $(ad-bc \neq 0)$

then $Tz = z \iff \dfrac{az+b}{cz+d} = z$

$$\iff cz^2 + (d-a)z - b = 0 \quad \text{—} (*)$$

If $c \neq 0$, $(*)$ has 1 or 2 roots

$\Rightarrow$ T has 1 or 2 fixed points (in $\mathbb{C} \subset \hat{\mathbb{C}}$)

(note: ∞ is not fixed as $T(\infty) = \frac{a}{c}$)

If $c = 0$, $(*)$ has 1 solution $z = \dfrac{b}{d-a}$ provided $a \neq d$.

Note that in this case, $T\infty = \infty$

$\Rightarrow$ T has 2 fixed points $\frac{b}{d-a}$ & ∞ in $\hat{\mathbb{C}}$

(provided $a \neq d$)

If $c=0$, $a=d$, then $ad-bc \neq 0 \Rightarrow a=d\neq 0$

and $Tz = \frac{az+b}{\cancel{cz+}d} = z + \frac{b}{d}$

has $\begin{cases} \text{unique fixed point } \infty, \text{ if } b \neq 0 \\ \text{infinitely many fixed points, if } b=0 \\ \quad (\text{in fact, } \forall z \in \hat{\mathbb{C}} \text{ \& } T = Id_{\hat{\mathbb{C}}}) \end{cases}$

We've proved that

Lemma:
- If T is not the identity transformation, then T has only 1 or 2 fixed points.
- A Möbius tranformation with 3 or more fixed points must be the identity.

Thm (The Fundamental Theorem of Möbius Geometry)

There is a unique Möbius transformation taking any 3 distinct (extended) complex numbers z_1, z_2, z_3 to any other 3 distinct (extended) complex numbers w_1, w_2, w_3.

(i.e. $\exists ! \; T \in \mathbb{M}$ s.t. $Tz_1 = w_1$, $Tz_2 = w_2$, $Tz_3 = w_3$)

The required Möbius transformation is given by

$$\boxed{\frac{w - w_2}{w - w_3} \cdot \frac{w_1 - w_3}{w_1 - w_2} = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}}$$

Pf (of the Fundamental Theorem of Möbius Geometry)

The formula

$$\frac{w - w_2}{w - w_3} \cdot \frac{w_1 - w_3}{w_1 - w_2} = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

provided can be seen from the following steps:

Step 1: $\forall\, z_1, z_2, z_3$ distinct (extended) cpx numbers, $\exists$ Möbius transformation T such that

$$Tz_1 = 1,\quad Tz_2 = 0,\quad Tz_3 = \infty \qquad \text{—— } (*)$$

Pf of Step 1:

$$\left.\begin{array}{l} Tz_2 = 0 \Rightarrow \dfrac{*(z - z_2)}{***} \\[2em] Tz_3 = \infty \Rightarrow \dfrac{***}{*(z - z_3)} \end{array}\right\}$$

$\Rightarrow$ Tz must be the form

$$Tz = \beta\, \frac{z - z_2}{z - z_3}$$

for some complex number β.

Then $Tz_1 = 1 \Rightarrow$ $1 = \beta \cdot \dfrac{z_1 - z_2}{z_1 - z_3}$

$$\Rightarrow \quad \beta = \frac{z_1 - z_3}{z_1 - z_2}$$

Hence

$$Tz = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

is the required Möbius transformation. $\#$

Step 2:

$$\begin{tikzcd}
z_1 \arrow[r, "T"] & 1 & w_1 \arrow[l, "U"'] \\
z_2 \arrow[r] & 0 & w_2 \arrow[l] \\
z_3 \arrow[r] \arrow[rr, bend right, "U^{-1}\circ T"'] & \infty & w_3 \arrow[l]
\end{tikzcd}$$

$\forall$ distinct z_1, z_2, z_3 & distinct w_1, w_2, w_3,

By step 1, $\exists\, T$ s.t. $\begin{cases} Tz_1 = 1 \\ Tz_2 = 0 \\ Tz_3 = \infty \end{cases}$

and U s.t. $\begin{cases} Uw_1 = 1 \\ Uw_2 = 0 \\ Uw_3 = \infty \end{cases}$

(T, U are Möbius transformations)

Then $S = U^{-1} \circ T$ is a Möbius Transformation such that $Sz_1 = U^{-1} \circ T(z_1) = U^{-1}(Tz_1)$

$$= U^{-1}(1) = w_1$$

Similarly $Sz_2 = w_2$ & $Sz_3 = w_3$.

Hence, we've proved that for any distinct z_1, z_2, z_3 and distinct w_1, w_2, w_3, $\exists$ a Möbius transformation S such that $S z_i = w_i$, $i = 1,2,3$ #

(Ex: Why this gives the formula?)

<u>Finally</u> (<u>Uniqueness</u>):

If U_1 & U_2 are Möbius transformations s.t.

$$U_k(z_i) = w_i, \quad i = 1,2,3; \quad k = 1,2$$

Then $U_1 = U_2$.

<u>Pf of Final step</u>:

Consider $U_2^{-1} \circ U_1$, then it is a Möbius transformation such that $U_2^{-1} \circ U_1 (z_i) = U_2^{-1}(w_i) = z_i$, $i = 1,2,3$.

$\Rightarrow$ $U_2^{-1} \circ U_1$ has at least 3 fixed points (as z_1, z_2, z_3 are distinct.)

By lemma (right before the Theorem)

$$\upsilon_2^{-1} \circ \upsilon_1 = Id_{\hat{\mathbb{C}}}$$

$$\Rightarrow \quad \upsilon_1 = \upsilon_2 \quad . \quad \#$$

$\boxed{\text{Corollary: All figures consisting of 3 distinct points are congruent in Möbius geometry.}}$

Remark: This corollary $\Rightarrow$ Möbius geometry is not isomorphic to Euclidean geometry, and Euclidean distance is not an invariant.