THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5010 Linear Analysis (Fall 2019)
Suggested Solution of Homework 3: p.125: 1, 3

1. Show that the following are continuous functionals,

   (a) $\phi \boldsymbol{x} := \sum_{n=1}^{\infty} \frac{1}{n} a_n$ on ℓ^2; (3 marks)

   (b) $\phi \boldsymbol{x} := \sum_{n=0}^{\infty} e^{inw} a_n$, and $\phi \boldsymbol{x} := \sum_{n=0}^{\infty} \sin(nw) a_n$ on ℓ^1 (w is a fixed real number); (3 marks)

   (c) $\delta_1 \boldsymbol{x} := a_1$ on $\ell^1, \ell^2, \ell^\infty$. (2 marks)

   Solution. For a functional, we mean a linear map from the normed space to its scalar field, that is usually real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Here we only show the continuity of these maps. The linearity can be checked directly.

   (a) We say that $\boldsymbol{x} = (a_0, a_1, \ldots, a_n, \ldots)$ is an element of ℓ^2 if

   $$\sum_{n=0}^{\infty} |a_n|^2 < \infty$$

   We put $\|\boldsymbol{x}\|_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2}$, which is the norm on ℓ^2. Assuming the linearity, we only need to show that there exists some constant $C > 0$ such that

   $$|\phi \boldsymbol{x}| \leq C \|\boldsymbol{x}\|_2 \quad \text{for every } \boldsymbol{x} \in \ell^2$$

   Let $\boldsymbol{x} = (a_0, a_1, \ldots, a_n, \ldots) \in \ell^2$.

   $$\begin{aligned} |\phi \boldsymbol{x}| &= \left| \sum_{n=1}^{\infty} \frac{1}{n} a_n \right| \\ &\leq \lim_{N \to \infty} \sum_{n=1}^{N} \left| \frac{1}{n} a_n \right| \\ &\leq \lim_{N \to \infty} \sqrt{\sum_{n=1}^{N} \left(\frac{1}{n} \right)^2} \sqrt{\sum_{n=1}^{N} |a_n|^2} \qquad \text{(Cauchy-Schwarz inequality)} \\ &\leq \sqrt{\sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^2} \|\boldsymbol{x}\|_2, \end{aligned}$$

   where $C = \sqrt{\sum_{n=1}^{\infty} (\frac{1}{n})^2}$ is a finite number.

   (b) For an element $\boldsymbol{x} = (a_0, a_1, \ldots, a_n, \ldots)$ in ℓ^1. Its norm is given by

   $$\|\boldsymbol{x}\|_1 = \sum_{n=0}^{\infty} |a_n|$$

Notice that when w is a real number, inw is an imaginary number for each $n \in \mathbb{N}$. Thus, $|e^{inw}| = e^{\mathrm{Re}(inw)} = e^0 = 1$. Now, we have

$$\begin{aligned} |\phi \boldsymbol{x}| &= \left| \sum_{n=0}^{\infty} e^{inw} a_n \right| \\ &\leq \sum_{n=0}^{\infty} |e^{inw} a_n| \\ &= \sum_{n=0}^{\infty} |a_n| = \|\boldsymbol{x}\|_1 \end{aligned}$$

Similar for $\phi \boldsymbol{x} := \sum_{n=0}^{\infty} \sin(nw) a_n$ on ℓ^1.

(c) For $\boldsymbol{x} = (a_0, a_1, \ldots, a_n, \ldots)$, we only need to observe that

$$|a_1| \leq \|\boldsymbol{x}\|_1 = \textstyle\sum_{n=0}^{\infty} |a_n|$$

$$|a_1| \leq \|\boldsymbol{x}\|_2 = \sqrt{\textstyle\sum_{n=0}^{\infty} |a_n|^2}$$

$$|a_1| \leq \|\boldsymbol{x}\|_\infty = \sup_{n \geq 0} |a_n|$$

3. The right-shift operator is defined by $R\boldsymbol{x} := (0, a_0, a_1, \ldots)$, where $\boldsymbol{x} := (a_0, a_1, \ldots)$. It is an operator that satisfies $\|R\boldsymbol{x}\| = \|\boldsymbol{x}\|$ both as $\ell^\infty \to \ell^\infty$ and $\ell^1 \to \ell^1$; it is 1-1 and its image is closed. Note that $LR = I \neq RL$. Show that it is also continuous as $R : \ell^1 \to \ell^\infty$. (2 marks)

Solution. Let $R : \ell^1 \to \ell^\infty$ be the right-shift operator. To see that R is continuous, we may show that

$$\|R\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_1$$

For $\boldsymbol{x} = (a_0, a_1, \ldots)$,

$$(R\boldsymbol{x})(k) = \begin{cases} 0, & \text{if } k = 0; \\ a_{k-1}, & \text{if } k \geq 1. \end{cases}$$

Hence, $\|R\boldsymbol{x}\|_\infty = \sup_{k \geq 0} |(R\boldsymbol{x})(k)| = \sup_{k \geq 0} |a_k|$.

For each $k \geq 0$, $|a_k| \leq \sum_{n=0}^{\infty} |a_n| = \|\boldsymbol{x}\|_1$, therefore $\|R\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_1$.