2019 Fall MMAT 5010 Linear Analysis

Solution 1

p.100: 1, 7

1. Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Which axiom does $\|\cdot\|_p$ fail when $p < 1$?

   Solution. Consider the vector space $\mathbb{K}^N$. Let $\mathbf{x} = (x_1, \ldots, x_N), \mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{K}^N$ and $\alpha \in \mathbb{K}$.

   We first show that $\|\mathbf{x}\|_1 := \sum_{i=1}^N |x_i|$ is a norm on $\mathbb{K}^N$.

   (i) Clearly $\|\mathbf{x}\|_1 \geq 0$. Moreover, $\|\mathbf{x}\|_1 = 0$ if and only if $x_i = 0$ for every $1 \leq i \leq N$, i.e. $\mathbf{x} = 0$.

   (ii) $\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^N |\alpha x_i| = |\alpha| \sum_{i=1}^N |x_i| = |\alpha|\, \|\mathbf{x}\|_1$.

   (iii) $\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^N |x_i + y_i| \leq \sum_{i=1}^N (|x_i| + |y_i|) = \sum_{i=1}^N |x_i| + \sum_{i=1}^N |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$

   Hence, $\|\cdot\|_1$ is a norm on $\mathbb{K}^N$.

   Next we show that $\|\mathbf{x}\|_\infty := \max_{1\leq i\leq N} |x_i|$ is a norm on $\mathbb{K}^N$.

   (i) Clearly $\|\mathbf{x}\|_\infty \geq 0$. Moreover, $\|\mathbf{x}\|_\infty = 0$ if and only if $x_i = 0$ for every $1 \leq i \leq N$, i.e. $\mathbf{x} = 0$.

   (ii) $\|\alpha\mathbf{x}\|_\infty = \max_{1\leq i\leq N} |\alpha x_i| = |\alpha| \max_{1\leq i\leq N} |x_i| = |\alpha|\, \|\mathbf{x}\|_\infty$, because $|\alpha| \geq 0$.

   (iii) $\|\mathbf{x}+\mathbf{y}\|_\infty = \max_{1\leq i\leq N}(|x_i + y_i|) \leq \max_{1\leq i\leq N}(|x_i| + |y_i|) \leq \max_{1\leq i\leq N} |x_i| + \max_{1\leq i\leq N} |y_i| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$.

   Hence, $\|\cdot\|_\infty$ is a norm on $\mathbb{K}^N$.

   We now show that $\|\cdot\|$ fails the triangle inequality, hence is not a norm, when $p < 1$. For example, consider $\mathbf{x} = (1/2, 0), \mathbf{y} = (0, 1/2) \in \mathbb{R}^2$. Then

   $$\|\mathbf{x}+\mathbf{y}\|_p = \sqrt[p]{\left(\frac{1}{2}\right)^p + \left(\frac{1}{2}\right)^p} = 2^{\frac{1}{p}-1}$$

   while

   $$\|x\|_p + \|y\|_p = \frac{1}{2} + \frac{1}{2} = 1$$

   When $p < 1$, we have $\frac{1}{p} - 1 > 0$ and hence $\|\mathbf{x}+\mathbf{y}\|_p = 2^{\frac{1}{p}-1} > 1 = \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$. The triangle inequality fails.

7. The norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are all equivalent on $\mathbb{R}^N$ since (prove!)

   $$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq N\|\mathbf{x}\|_\infty.$$

   But they are not equivalent for sequences or functions! Find sequences of functions that converge in $L^1[0,1]$ but not in $L^\infty[0,1]$, or vice-versa. Can sequences converge in ℓ^1 but not in ℓ^∞?

   Solution.

(a) Let $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$. For each fixed $1 \le i \le N$, from the inequality

$$|x_i|^2 \le \sum_{j=1}^{N} |x_j|^2,$$

and taking square root, we have $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$. Note also

$$\left(\sum_{j=1}^{N} |x_j|\right)^2 = \sum_{j=1}^{N} |x_j|^2 + 2\sum_{i<j} |x_i x_j| \ge \sum_{j=1}^{N} |x_j|^2,$$

it is easy to see that $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. Finally, note that $|x_j| \le \|\mathbf{x}\|_\infty$ for each $1 \le j \le N$. Summing up when j goes from 1 to N gives $\|\mathbf{x}\|_1 \le N\|\mathbf{x}\|_\infty$.

(b) Recall that $L^1[0,1] := \{f : A \to \mathbb{C} : \int_0^1 |f(x)|\,\mathrm{d}x < \infty\}$ with norm defined by

$$\|f\|_{L^1} := \int_0^1 |f(x)|\,\mathrm{d}x.$$

while $L^\infty[0,1] := \{f : [0,1] \to \mathbb{C} : f \text{ is measurable AND } \exists c\, |f(x)| \le c \text{ a.e. } x\}$ with norm defined by

$$\|f\|_{L^\infty} := \sup_{x \text{ a.e.}} |f(x)|$$

Consider the sequence (f_n) of functions on $[0,1]$, defined by $f_n = \chi_{[0,\frac{1}{n}]}$, i.e.

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{n}]; \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\|f_n\|_{L^1} = \int_0^{\frac{1}{n}} \mathrm{d}x = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This means that f_n converges to 0 in $L^1[0,1]$. However, (f_n) is not a Cauchy sequence in $L^\infty[0,1]$, and hence not a convergent sequence: for $n > m$, we have

$$\|f_n - f_m\|_{L^\infty} = \sup_{x \text{ a.e.}} \left|\chi_{(\frac{1}{n},\frac{1}{m}]}(x)\right| = 1.$$

Finally, we note that for every function f on $[0,1]$,

$$\|f\|_{L^1} = \int_0^1 |f(x)|\,\mathrm{d}x \le \int_0^1 \|f\|_{L^\infty}\,\mathrm{d}x = \|f\|_{L^\infty}.$$

If (f_n) is a sequence in $L^\infty[0,1]$ and converges to the limit f in L^∞-norm, then f_n converges to f in L^1-norm.

(c) Note that for every $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \ell^1$,

$$\|\mathbf{y}\|_{\ell^\infty} = \sup_{i \in \mathbb{N}} |y_i| \le \sum_{j=1}^{\infty} |y_j| = \|\mathbf{y}\|_{\ell^1}.$$

In particular, $\mathbf{y} \in \ell^\infty$. Suppose a sequence $(\mathbf{x_n})$ converges to $\mathbf{x}$ in ℓ^1. Then,

$$\|\mathbf{x_n} - \mathbf{x}\|_{\ell^\infty} \le \|\mathbf{x_n} - \mathbf{x}\|_{\ell^1} \to 0 \text{ as } n \to \infty.$$

That is, $(\mathbf{x_n})$ converges to the same limit $\mathbf{x}$ in ℓ^∞.

p. 106: 3

3. When X, Y are Banach spaces over the same field, so is $X \times Y$ (Proposition 4.7).

   Solution. Recall that $X \times Y$ is a normed space with norm given by

$$\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y.$$

   It suffices to show that if X and Y are complete, then so is $X \times Y$. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in $X \times Y$. Note that

$$\|(x_n, y_n) - (x_m, y_m)\|_{X\times Y} = \|x_n - x_m\|_X + \|y_n - y_m\|_Y \geq \|x_n - x_m\|_X.$$

   Since LHS converges to 0 as $n, m \to \infty$, we have $\|x_n - x_m\|_X \to 0$. It shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space X. It therefore converges to some point $x \in X$. By similar reasoning, the sequence $\{y_n\}$ also converges to some $y \in Y$. Consequently,

$$\|(x_n, y_n) - (x, y)\|_{X\times Y} = \|x_n - x\|_X + \|y_n - y\|_Y \to 0 \quad \text{as } n \to \infty,$$

   This says that the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ converges to (x, y) in $X \times Y$. Thus, $X \times Y$ is complete.