MMAT5010 2021 Assignment 6

Q1. Let $a, b \in X$, $a \neq b$. Using Hahn-Banach Theorem there exists $x^* \in X^*$ such that $x^*(a) \neq x^*(b)$. Fix some element $0 \neq y \in Y$. Define $T : X \to Y$ by $T(x) = x^*(x)y$. Then T is linear because

$$T(c_1x_1 + c_2x_2) = (x^*(c_1x_1 + c_2x_2))y = c_1x^*(x_1)y + c_2x^*(x_2)y = c_1T(x_1) + c_2T(x_2).$$

Moreover, T is bounded because

$$||T(x)|| = ||x^*(x)y|| \leq ||x^*||\,||y||\,||x||$$

Hence T is the required operator.

Q2. The element a is $(2, -3)$. We have $||a||_\infty = 3$. It remains to verify $||f|| = 3$.

For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$|f(x_1, x_2)| \leq 2|x_1| + 3|x_2| \leq 3||(x_1, x_2)||_1$$

Therefore $||f|| \leq 3$, and $||f|| = 3$ because $f(0, -1) = 3$. (Note $(0, -1)$ has $||\cdot||_1$-norm 1)