MMAT5010 2021 Assignment 5

Q1. (i): (**Method 1**) Note that

$$|\delta_n(x)| = |x(n)| \le \sup_{n=1,2,\dots} |x(n)| = ||x||$$

we have $\delta_n \in c_0^*$ and $||\delta_n|| \le 1$. Moreover, $||\delta_n|| = 1$ becuase $\delta_n(x_n) = 1$ where $x_n = (0, 0, \dots, 0, 1, 0, \dots)$.

(**Method 2**)Recall that $c_0^* = \ell_1$. Under this observation, $\delta_n = e_n \in \ell_1$ and $||\delta_n||_{c_0^*} = ||e_n||_{\ell_1} = 1$.

(ii): If $x \in c_0$, then $\lim_{n\to\infty} x_n = 0$. This is equivalent to saying that $\lim_{n\to\infty} \delta_n(x) = 0$. By ($i$), $||\delta_n|| = 1$, hence δ_n cannot converge to 0.

Q2. We will prove that $||\varphi|| = 2$.

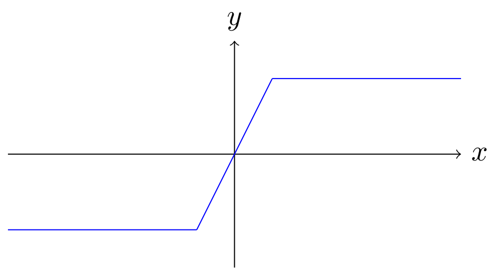
First we must show $|\varphi(f)| \le 2||f||$ for all $f \in X = C[-1, 1]$. This can be directly calculated:

$$|\varphi(f)| \le \left| \int_0^1 f(x)\,dx \right| + \left| \int_{-1}^0 f(x)\,dx \right| \le ||f|| + ||f||$$

Next we must show $||\varphi|| = 2$. We will construct a sequence (f_n), $f_n \in X$, $||f_n|| = 1$ and $\lim_{n\to\infty} \varphi(f_n) = 2$. Define f_n by the following:

- $f_n = 1$ on $(\frac{1}{n}, 1]$
- $f_n = -1$ on $[-1, -\frac{1}{n})$
- f_n is linear on $[-\frac{1}{n}, \frac{1}{n}]$ and $f_n(-\frac{1}{n}) = -1$, $f_n(\frac{1}{n}) = 1$.

The graph of f_n looks like:



It can be checked that each f_n is indeed continuous on $[-1, 1]$ and $||f_n|| = 1$. Moreover, $\varphi(f_n) = 2 - \frac{1}{n}$. Hence $||\varphi|| = 2$.