MMAT5010 2021 Assignment 2

Q1. Recall the definition of ℓ_1:

$$\ell_1 = \left\{ (a_n)_{n=1}^{\infty} : a_n \in \mathbb{R} \text{ and } \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

To show that c_{00} is not a closed subspace of ℓ_1, it needs to find an element $x \in \ell_1$, $x \notin c_{00}$ and a sequence $(x_n)_{n=1}^{\infty}$, $x_n \in c_{00}$, $x_n \to x$.

Note $x = (1, 1/2, 1/2^2, 1/2^3, \dots)$ clearly lies in ℓ_1 but not in c_{00}. Define $x_n = (1, 1/2, \dots, 1/2^n, 0, 0, \dots) \in c_{00}$. We check that $x_n \to x$:

$$||x_n - x||_{\ell_1} = \sum_{i>n} \frac{1}{2^i} = \frac{1}{2^n} \to 0$$

Q2. (i) If $f : X \to Y$ is continuous, to check that $G(f)$ is closed (in $(X \times Y, ||\cdot||_0)$), it needs to show: for every sequence $(z_n)_{n=1}^{\infty}$, $z_n \in G(f)$, if $z_n \to z \in X \times Y$ in $||\cdot||_0$, then $z \in G(f)$.

Let $z_n \in G(f)$, $z_n \to z \in X \times Y$. Since $z_n \in G(f)$, there exists $x_n \in X$ such that $z_n = (x_n, f(x_n))$. Also, there exists $x \in X$, $y \in Y$ such that $z = (x, y)$. Then

$$||x_n - x|| \le ||z_n - z||_0$$

So $x_n \to x$, by continuity of f, $f(x_n) \to f(x)$. It follows that $z_n \to (x, f(x))$, but since limit is unique, we must have $f(x) = y$. Hence $z = (x, y) \in G(f)$.

(ii) Note if we define $z_n = (-1/n, 0)$, then $z_n \in G(f)$. Moreover $z_n \to (0, 0)$ in $||\cdot||_0$ norm, but $(0, 0) \notin G(f)$. Hence $G(f)$ is not closed.