Solution 6

Hand in no. 1, 3, 4 and 5 by March 28.

Consider the initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = u_{xx} + F(x,t), & \text{in } [0,\pi]\times(0,\infty), \\ u(x,0) = f(x), & \text{in } [0,\pi], \\ u(0,t) = g_1(t),\ u(\pi,t) = g_2(t), & t>0. \end{cases} \tag{1}$$

1. Find the solution of (1) when $F \equiv 0$, $g_1 = 5$ and $g_2 \equiv 0$. Hint: Find φ so that $v = u - \varphi$ satisfies (1) with F, g_1, g_2 all vanish.

   Solution. Let
   $$\varphi(x) = \varphi(x,t) = \left(1 - \frac{x}{\pi}\right) g_1(t) + \frac{x}{\pi} g_2(t) = 5\left(1 - \frac{x}{\pi}\right).$$
   It is easy to check that
   $$\varphi_t = \varphi_{xx} = 0, \quad \varphi(0) = 5 \text{ and } \quad \varphi(\pi) = 0.$$
   By Theorem 4, there is a solution v of (1) with F, g_1, g_2 all vanish and $v(x,0) = f(x) - \varphi(x)$. Now $u \equiv v + \varphi$ is the required solution.

2. Find the solution of (1) when g_1, g_2 vanish and $F(x,t) = C$. Hint: Consider the function w satisfying $w'' + C = 0$, $w(0) = w(\pi) = 0$.

   Solution. Clearly, $w(x) := -\frac{C}{2}x(x-\pi)$ satisfies $w'' + C = 0$ and $w(0) = w(\pi) = 0$. By Theorem 4, there is a solution v of (1) with F, g_1, g_2 all vanish and $v(x,0) = f(x) - w(x)$. Now $u \equiv v + w$ is the required solution.

3. Find the solution of (1) when g_1, g_2 vanish and F is nice. Hint: Make use of the Fourier expansion of $F(x,t) = \sum F_n(t) \sin nx$.

   Solution. As in the proof of Theorem 4, write
   $$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx,$$
   and
   $$f(x,t) = \sum_{n=1}^{\infty} B_n \sin nx.$$
   Assume that F is sufficiently smooth and satisfies $F(0,t) = F(\pi,t) = 0$. Then $F(x,t)$ admits a sine series representation:
   $$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin nx.$$
   In order $u(x,t)$ represents a solution to (1), we require
   $$u_t - u_{xx} - F(x,t) = \sum_{n=1}^{\infty} (b_n'(t) + n^2 b_n(t) - F_n(t)) \sin nx = 0.$$

Thus we need to solve

$$b_n'(t) + n^2 b_n(t) - F_n(t), \qquad b_n(0) = B_n.$$

By multiplying an integrating factor $e^{n^2 t}$ on both sides, the above ODE can be solved with

$$b_n(t) = \left(B_n + \int_0^t F_n(s) e^{n^2 s} ds \right) e^{-n^2 t}.$$

A solution to (1) is thus

$$\sum_{n=1}^{\infty} \left(B_n + \int_0^t F_n(s) e^{n^2 s} ds \right) e^{-n^2 t} \sin nx. \tag{2}$$

4. Find the solution of (1) when g_1, g_2 vanish and $F(x,t) = e^{-t} \sin x$.

   Solution. By question 3, we have

$$b_1(t) = \left(B_1 + \int_0^t e^{-s} e^{s} ds \right) e^{-t} = (B_1 + t) e^{-t},$$

   and for $n \geq 2$,

$$b_n(t) = B_n.$$

   Hence the solution to (1) is

$$u(x,t) = te^{-t} \sin x + \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx.$$

Consider the initial-boundary value problem for the wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ c > 0 & \text{in } [0,\pi] \times (0,\infty), \\ u(x,0) = f(x), \ u_t(x,0) = g(x), & \text{in } [0,\pi], \\ u(0,t) = u(\pi,t) = 0, & t > 0. \end{cases} \tag{3}$$

5. Let $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$ be the solution to (3).

   (a) Show that b_n satisfies the differential equation

$$b_n''(t) + n^2 c^2 b_n(t) = 0.$$

   (b) Show that the solution u is given by

$$u(x,t) = \sum_{n=1}^{\infty} (c_n \cos nct + d_n \sin nct) \sin nx,$$

   where c_n and d_n are determined by

$$f(x) \sim \sum_{n=1}^{\infty} c_n \sin nx,$$

   and

$$g(x) \sim \sum_{n=1}^{\infty} cnd_n \sin nx,$$

Solution.

(a) Let

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin nx \quad \text{and} \quad u(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin nx.$$

Then

$$v_n(t) = \frac{2}{\pi} \int_0^{\pi} u_{tt}(x,t) \sin nx \, dx = \frac{d^2}{dt^2} \frac{2}{\pi} \int_0^{\pi} u(x,t) \sin nx \, dx = b_n''(t).$$

By integration by parts (twice), we have

$$\begin{aligned} w_n(t) &= \frac{2}{\pi} \int_0^{\pi} u_{xx}(x,t) \sin nx \, dx \\ &= \frac{2}{\pi} u_x \sin nx \Big|_0^{\pi} - n \frac{2}{\pi} \int_0^{\pi} u_x \cos nx \, dx \\ &= -\frac{2n}{\pi} u \cos nx \Big|_0^{\pi} - \frac{2n^2}{\pi} \int_0^{\pi} u \sin nx \, dx \\ &= -n^2 b_n(t). \end{aligned}$$

Now

$$b_n''(t) + n^2 c^2 b_n(t) = v_n(t) - c^2 w_n(t) = \frac{2}{\pi} \int_0^{\pi} (u_{tt} - c^2 u_{xx}) dx = 0.$$

(b) Solving the ODE $b_n''(t) + n^2 c^2 b_n(t) = 0$, we have

$$b_n(t) = A_n \cos nct + B_n \sin nct,$$

where A_n, B_n are constants. The initial conditions imply that

$$b_n(0) = \frac{2}{\pi} \int_0^{\pi} u(x,0) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = c_n,$$

and

$$b_n'(0) = \frac{2}{\pi} \int_0^{\pi} u_t(x,0) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx \, dx = cnd_n.$$

Thus $A_n(0) = b_n(0) = c_n$ and $B_n(0) = \dfrac{b_n'(0)}{nc} = d_n$. Hence, the solution u is given by

$$u(x,t) = \sum_{n=1}^{\infty} (c_n \cos nct + d_n \sin nct) \sin nx.$$