Solution 5

p. 177: 1, 9, 15

1. If $T, S : X \to Y$ are linear maps on inner product spaces such that $\langle y, Tx\rangle = \langle y, Sx\rangle$ for all $x \in X, y \in Y$, then $T = S$. Example 10.7(3) is false for real spaces: Find a non-zero 2×2 real matrix T such that $\langle \mathbf{x}, T\mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathbb{R}^2$.

   Solution. By linearity of inner product, we have

   $$\langle y, (T - S)x\rangle = 0 \quad \text{for all } x \in X, y \in Y.$$

   By taking $y = (T - S)x$, we have

   $$\|(T - S)x\|^2 = \langle (T - S)x, (T - S)x\rangle = 0 \quad \text{for all } x \in X.$$

   Thus $T - S = 0$, that is $T = S$.

   To see that Example 10.7(3) is false for real spaces, consider the non-zero matrix

   $$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Then for all $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$,

   $$\begin{aligned} \langle \mathbf{x}, T\mathbf{x}\rangle &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_2 & x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= -x_2x_1 + x_1x_2 \\ &= 0. \end{aligned}$$

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9. The 1-norm and ∞-norm defined on $\mathbb{R}^2$ do not come from inner product. Find two vectors that do not satisfy the parallelogram law.

   Solution. Let $\mathbf{x} = (1, 0), \mathbf{y} = (0, 1)$. Then

   $$2\|\mathbf{x}\|_1^2 + 2\|\mathbf{y}\|_1^2 = 2(1 + 0)^2 + 2(0 + 1)^2 = 4,$$

   while

   $$\|\mathbf{x} + \mathbf{y}\|_1^2 + \|\mathbf{x} - \mathbf{y}\|_1^2 = (1 + 1)^2 + (1 + 1)^2 = 8.$$

   On the other hand,

   $$2\|\mathbf{x}\|_\infty^2 + 2\|\mathbf{y}\|_\infty^2 = 2(1)^2 + 2(1)^2 = 4,$$

   while

   $$\|\mathbf{x} + \mathbf{y}\|_\infty^2 + \|\mathbf{x} - \mathbf{y}\|_\infty^2 = (1)^2 + (1)^2 = 2.$$

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15. Let $d := d(x, [\![y]\!]) = \inf_\lambda \|x + \lambda y\|$, where y is a unit vector; show that

   (a) $d = \|x + \lambda_0 y\|$ for some λ_0,

   (b) $|\langle x, y\rangle|^2 = \|x\|^2 - d^2$, and

   (c) $y \perp (x + \lambda_0 y)$. (Rather than $y \perp (x - \lambda_0 y)$.)

Solution. (a) Write $\lambda = a + bi$ and let $f(a, b) = \|x + (a + bi)y\|^2$. Then

$$\begin{aligned} f(a,b) &= \langle x + (a+bi)y, x + (a+bi)y\rangle \\ &= \langle x, x\rangle + (a+bi)\langle x, y\rangle + (a-bi)\langle y, x\rangle + (a^2+b^2)\langle y, y\rangle \\ &= \|x\|^2 + (a+bi)\langle x, y\rangle + (a-bi)\langle y, x\rangle + (a^2+b^2) \end{aligned}$$

since $\|y\| = 1$. Note that f is a differentiable real-valued function. Now

$$\frac{\partial f}{\partial a} = \langle x, y\rangle + \langle y, x\rangle + 2a$$

while

$$\frac{\partial f}{\partial b} = i\langle x, y\rangle - i\langle y, x\rangle + 2b.$$

Thus f attains its minimum at (a_0, b_0) where

$$a_0 = -\frac{\langle x, y\rangle + \langle y, x\rangle}{2}, \quad b_0 = -\frac{i\langle x, y\rangle - i\langle y, x\rangle}{2}.$$

Therefore, for $\lambda_0 := a_0 + b_0 i = -\langle y, x\rangle$, $\|x + \lambda_0 y\| = \inf_\lambda \|x + \lambda y\| = d$.

(b) By direct computation,

$$\begin{aligned} d^2 = \|x + \lambda_0 y\|^2 &= \langle x + \lambda_0 y, x + \lambda_0 y\rangle \\ &= \langle x, x\rangle + (-\langle y, x\rangle)\langle x, y\rangle + \overline{(-\langle y, x\rangle)}\langle y, x\rangle + |-\langle y, x\rangle|^2\langle y, y\rangle \\ &= \langle x, x\rangle - \langle y, x\rangle\overline{\langle y, x\rangle} - \overline{\langle y, x\rangle}\langle y, x\rangle + |\langle y, x\rangle|^2 \\ &= \|x\|^2 - |\langle y, x\rangle|^2 \\ &= \|x\|^2 - |\langle x, y\rangle|^2. \end{aligned}$$

(c) Since

$$\begin{aligned} \langle y, x + \lambda_0 y\rangle &= \langle y, x\rangle + \lambda_0\langle y, y\rangle \\ &= \langle y, x\rangle - \langle y, x\rangle\langle y, y\rangle \\ &= 0, \end{aligned}$$

we have $y \perp (x + \lambda_0 y)$.

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