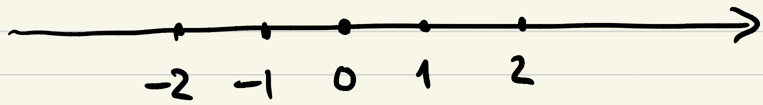
Jan. 20 :

e.g. 3 : Random walk



State space : $S = \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

$X_n \overset{\text{def.}}{=}$ the location valued in S on the real line

$n = 0, 1, 2, \cdots$

Requirement :

* From X_n to X_{n+1}, you walk by ξ_{n+1} steps. $\Rightarrow X_{n+1} = X_n + \xi_{n+1}$

* $\xi_1, \xi_2, \cdots$: i.i.d. , assume :

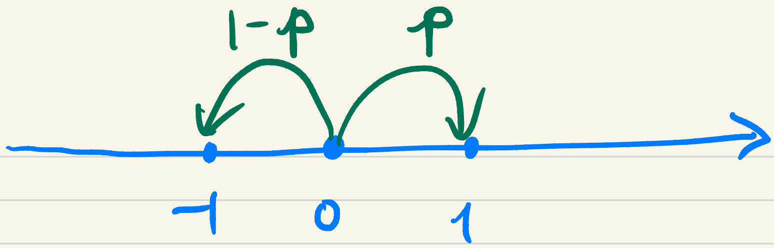
$$P(\xi_i = k) = f(k), \quad \forall k \in S$$

$\uparrow$ p.d.f. for each ξ_i

Q.: Find transition matrix.

$$\begin{aligned} P(x,y) &= P(X_{n+1} = y \mid X_n = x) \\ &= P(X_n + \xi_{n+1} = y \mid X_n = x) \\ &= P(\xi_{n+1} = y - x \mid X_n = x) \\ &= P(\xi_{n+1} = y - x) \\ &= f(y-x), \qquad \forall x, y \in S = \mathbb{Z} \end{aligned}$$

Remark : Simple random walk



pdf of each ξ_i : $\xi_i = \begin{cases} +1 & \text{with prob.} = p \\ -1 & \text{with prob} = 1-p \end{cases}$

means that you walk by exactly one step either to the right with prob. p or to the left with prob. $1-p$.

An interesting question:

Assume $X_0 = 0$, what happens to your location as $n \to \infty$?

e.g. 4 (Gambler's ruin chain)

- A player against the house.
- \$1 bet < win \$1 with p / lose \$1 with $1-p$
- start with a certain amount; ruined (quit the game) if your amount $= 0$

$X_n = \$$ at the n^{th} stage

$n = 0, 1, 2, \cdots$

$$S = \{0, 1, 2, \cdots\}$$

Q: transition probabilities

$$P(x, y) = P(X_{n+1} = y \mid X_n = x),$$
$$x, y \in S$$

If $x = 0$,

$$P(0, y) = \begin{cases} 1 & \text{for } y = 0 \\ 0 & \text{otherwise (for } y \geq 1) \end{cases}$$

Def. If $P(a, a) = 1$ (or $P(a, x) = 0$ for any $x \in S$ with $x \neq a$), then the state $a \in S$ is called an absorbing state.

If $x \geq 1$ (so, x is non-absorbing)

$$P(x, y) = P(X_{n+1} = y \mid X_n = \underset{\geq 1}{x})$$

$$= \begin{cases} p & \text{if } y = x+1 \text{ (win \$1)} \\ 1-p & \text{if } y = x-1 \text{ (lose \$1)} \\ 0 & \text{otherwise} \end{cases}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & \cdots & & \\ 1-p & 0 & p & \cdots & & \\ 0 & 1-p & 0 & p & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \end{array}\right] \end{array}$$

Slight modification : Add one more rule :

(*) If you reached the amount $= N$, you also quit the game

$X_0, X_1, \cdots$

or

$\{X_n\}_{n=0}^{\infty}$

$S = \{0, 1, \cdots, N\}$

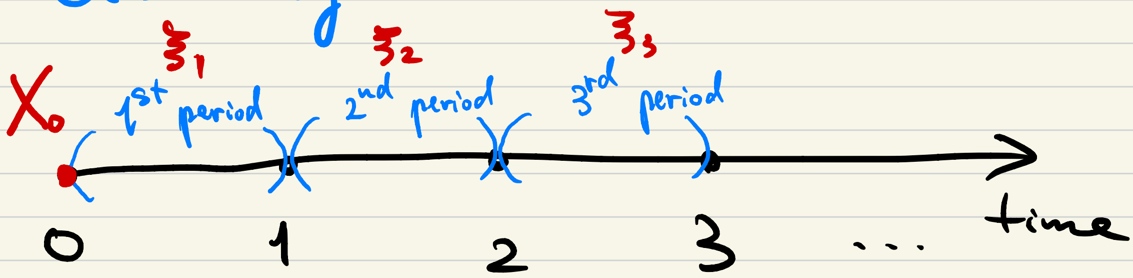
$P(0,0) = 1, \quad P(N, N) = 1$

both 0 & N are absorbing states

transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 1-p & 0 & p & \ddots & & & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & 0 & & \ddots & & \\ & & & & 1-p & 0 & p \\ 0 & \cdots & & \cdots & & 0 & 1 \end{array}\right] \end{array}$$

e.g. 5 Queueing chain



$X_0, X_1, X_2, \cdots$

$X_n \stackrel{\text{def.}}{=}$ no. of persons on the line waiting for the service at the time slot $n = 0, 1, 2, \cdots$

Requirement:

- ξ_n = no of arrivals in the n^{th} period

  $\xi_1, \xi_2, \xi_3, \cdots$: i.i.d.

  pdf = f, i.e.

  $$P(\xi_i = k) = f(k), \quad k = 0, 1, 2, \cdots$$

  A typical example for f is the Poisson distribution

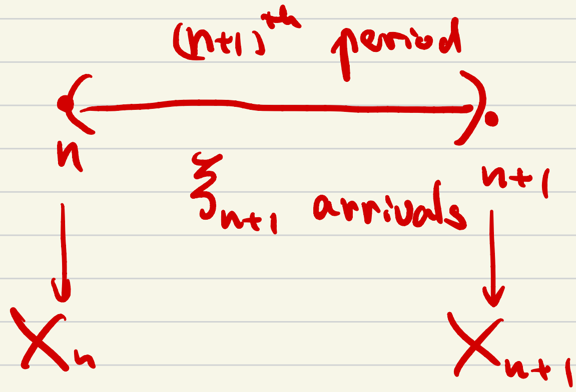
  $$f(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

- In each period $(n-1, n)$ for $n = 1, 2, \cdots$, exactly one person on the line will be served and leaves away.

then

$$X_{n+1} = \begin{cases} 0+\xi_{n+1} = \xi_{n+1} & \text{if } X_n = 0 \\ X_n + \xi_{n+1} - 1 & \text{if } X_n \geq 1 \end{cases}$$

$n = 0, 1, 2, \dots$



Q. : transition probabilities :

State space : $S = \{0, 1, 2, \dots\}$

$$P(x,y) = P(X_{n+1} = y \mid X_n = x)$$

$x, y \in S$

$$= \begin{cases} \text{if } x = 0, \text{ then } = P(\xi_{n+1} = y \mid X_n = x) \\ \qquad = P(\xi_{n+1} = y) \\ \qquad = f(y) \\ \text{if } x \geq 1, \text{ then } = P(\underbrace{X_n}_{=x} + \xi_{n+1} - 1 = y \mid X_n = x) \\ \qquad = P(\xi_{n+1} = y - x + 1 \mid X_n = x) \end{cases}$$

$x + \xi_{n+1} - 1 = y$

$\Leftrightarrow \xi_{n+1} = y - x + 1$

$$= P(\xi_{n+1} = \underbrace{y-x+1}_{\geq 0})$$

$$= f(y-x+1) \iff \boxed{y \geq x-1}$$

e.g. $f(k) = \begin{cases} e^{-\lambda} \dfrac{\lambda^k}{k!} & k=0,1,2,\cdots \\ 0, & \boxed{k \leq -1} \end{cases}$ Poisson

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0 & f(0) = e^{-\lambda} & f(1) = e^{-\lambda}\frac{\lambda}{1!} & f(2) = e^{-\lambda}\frac{\lambda^2}{2!} & \cdots \\ 1 & f(0-1+1) = f(0) = e^{-\lambda} & f(1-1+1) = f(1) = e^{-\lambda}\frac{\lambda}{1!} & f(2) = e^{-\lambda}\frac{\lambda^2}{2!} & \cdots \\ 2 & f(0-2+1) = f(-1) = 0 & f(1-2+1) = f(0) = e^{-\lambda} & e^{-\lambda}\frac{\lambda}{1!} & e^{-\lambda}\frac{\lambda^2}{2!} \end{array}$$

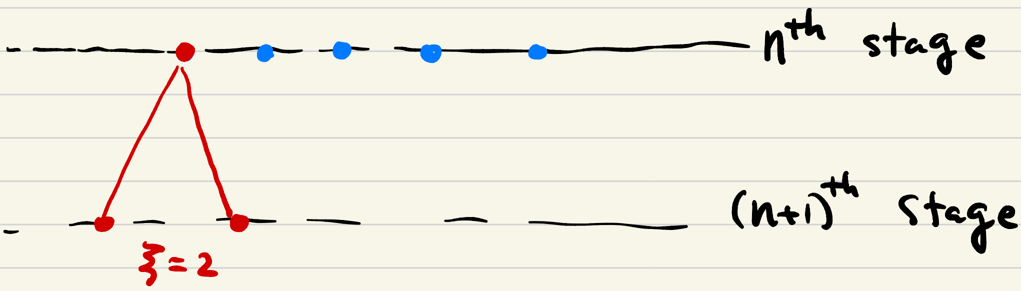
$\uparrow$ transition matrix

$$= \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ \end{array} \begin{bmatrix} e^{-\lambda} & e^{-\lambda}\frac{\lambda}{1!} & e^{-\lambda}\frac{\lambda^2}{2!} & \\ e^{-\lambda} & e^{-\lambda}\frac{\lambda}{1!} & e^{-\lambda}\frac{\lambda^2}{2!} & \\ & e^{-\lambda} & e^{-\lambda}\frac{\lambda}{1!} & e^{-\lambda}\frac{\lambda^2}{2!} \\ 0 & & \ddots & \ddots \end{bmatrix}$$

e.g. Branching Chain:

describe the population of offsprings

Require: Each individual in one generation generates ξ offsprings in the next generation r.v. independently.



$$X_0, \ X_1, \ X_2, \ \cdots, \ X_n, \ \cdots$$

$X_n \stackrel{\text{def.}}{=}$ total no. of offsprings at the n^{th} stage.

State space: $\mathcal{S} = \{0, 1, 2, \cdots\}$

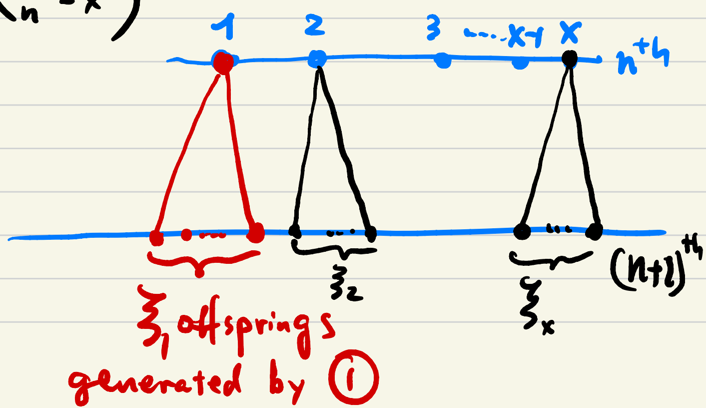
Transition probability:

$$P(X_{n+1} = y \mid X_n = x)$$

$$= P(\xi_1 + \xi_2 + \cdots + \xi_x = y \mid X_n = x)$$

$$= P(\xi_1 + \xi_2 + \cdots + \xi_x = y)$$

$\forall x, y = 0, 1, \cdots$



$\Downarrow$ Given that $X_n = x$

$X_{n+1} = y$

$\Leftrightarrow \xi_1 + \xi_2 + \cdots + \xi_x = y$