§3. (theoretical part, but important)

Existence & Uniqueness of SD

Def.

- $x \in S$ is positive recurrent if

  x is recurrent & $m_x = E_x(T_x) < \infty$

  (----)

- $x \in S$ is null recurrent if

  x is recurrent & $m_x = E_x(T_x) = \infty$

  (waiting time to return to x is infinite)

Remark. Recall

$$x \text{ recurrent} \iff P_x(N(x)=\infty) = 1$$
$$\iff E_x(N(x)) = \infty$$

- for null recurrent x,

By previous thm

$$\lim_{n\to\infty} E_x\left(\frac{N_n(x)}{n}\right) \overset{\downarrow}{=} \frac{1}{m_x} \underset{\uparrow}{=} \frac{1}{\infty} = 0$$

x is null recurrent

i.e. visit frequency is zero

(or the waiting return time is ∞)

- for (+)-recurrent x,

$$\lim_{n\to\infty} E_x\left(\frac{N_n(x)}{n}\right) = \frac{1}{m_x} \in (0, \infty)$$

positive finite

#

Thm 1 :

$$\left. \begin{array}{c} \underset{\text{(+) recurrent}}{x} \xrightarrow{\rho_{xy}>0} y \end{array} \right\} \Rightarrow y \text{ is (+)-recurrent.}$$

Proof.

$$\underset{\text{recurrent}}{x} \underset{\rho_{yx}>0}{\overset{\rho_{xy}>0}{\rightleftarrows}} \underset{\text{recurrent}}{y}$$

$\exists\, n_1 \geq 1$ s.t. $P^{n_1}(x,y) > 0$

$\exists\, n_2 \geq 1$ s.t. $P^{n_2}(y,x) > 0$

$\forall\, m \geq 1$:

$$P^{n_2+m+n_1}(y,y)$$

$$= (P^{n_2} \cdot P^m \cdot P^{n_1})(y,y)$$

$$\geq P^{n_2}(y,x)\, P^m(x,x)\, P^{n_1}(x,y)$$

$[n_2+1+n_1,\ n_2+n+n_1]$

$$\frac{\sum_{m=1}^{n}(\cdots)}{n} \Rightarrow \boxed{\frac{\sum_{m=1}^{n} P^{n_2+m+n_1}(y,y)}{n}} \geq P^{n_2}(y,x)\, \frac{\sum_{m=1}^{n} P^m(x,x)}{n}\, P^{n_1}(x,y)$$

$$\| $$

$$\frac{\sum_{m=1}^{n_2+n+n_1} P^m(y,y) - \sum_{m=1}^{n_2+n_1} P^m(y,y)}{n}$$

i.e.

$$E_y\left(\frac{N_{n_2+n+n_1}(y)}{n}\right) - E_y\left(\frac{N_{n_2+n_1}(y)}{n}\right)$$

$$\cancel{n_2+n+n_1} \times \frac{\cancel{n_2+n+n_1}}{n} \to 0$$

$$\geq P^{n_2}(y,x)\, E_x\left(\frac{N_n(x)}{n}\right) P^{n_1}(x,y)$$

let $n \to \infty$

$$\frac{1}{m_y} = \frac{1}{m_y} - 0 \geq \underbrace{P^{n_2}(y,x)}_{\substack{>0 \\ \text{finite}}} \times \underbrace{\frac{1}{m_x}}_{\substack{\in(0,\infty) \\ (\because x \text{ is } +\text{-recurrent})}} \times \underbrace{P^{n_1}(x,y)}_{\substack{>0 \\ \text{finite}}}$$

$\therefore \quad m_y < \infty$

$\therefore \quad y$ is (+)-recurrent. #

Thm 2. An irreducible MC with finite states is (+)-recurrent

(i.e. all states must be (+)-recurrent)

Proof. We see: all states are recurrent.

to show: all are also (+)-recurrent.

Otherwise, by thm 1, no (+)-recurrent states i.e. all states are null recurrent.

$$1 = \sum_{y \in S} P^m(x, y), \quad \forall x \in S$$

(P^m is a Markov matrix)

$$\frac{\sum_{m=1}^{n}(\dots)}{n} \Rightarrow 1 = \sum_{y \in S} E_x\left(\frac{N_n(y)}{n}\right), \quad \forall n \geq 1$$

$$\Rightarrow 1 = \lim_{n \to \infty} \sum_{y \in S} E_x\left(\frac{N_n(y)}{n}\right)$$

(finite)

$$= \sum_{y \in S} \lim_{n \to \infty} E_x\left(\frac{N_n(y)}{n}\right)$$

$$= \sum_{y \in S} \frac{1}{m_y}$$

$$= \sum_{y \in S} 0 \qquad (\because m_y = \infty, \forall y)$$

null recurr.

$$= 0 \;!!!$$ contradiction

#

Thm 3.

An irreducible (+)-recurrent MC has a unique SD π, given by

$$\pi(x) = \frac{1}{m_x}, \quad x \in S.$$

where $m_x = E_x(T_x) < \infty$. #

March 15:

Recall: Irreducible & recurrent MC:

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(x,y)}{n} = \frac{1}{m_y}, \quad \forall x, y$$

Proof of Thm 3:

Step 1. to show "!": namely

If $\pi = (\pi(x))_{x\in S}$ is a SD,
then $\pi(x) = \frac{1}{m_x}$, $\forall x \in S$

Indeed,

$$\pi = \pi P^m, \quad \forall m \geq 1$$

$$\pi(x) = \sum_{z\in S} \pi(z) P^m(z,x), \quad \forall x\in S, \ \forall m \geq 1$$

$$\frac{\sum_{m=1}^{n}(\cdots)}{n} \Rightarrow \pi(x) = \sum_{z\in S} \pi(z) \frac{\sum_{m=1}^{n} P^m(z,x)}{n}$$

$$n\to\infty \Rightarrow \pi(x) = \lim_{n\to\infty} \sum_{z\in S} \cdots$$

↳ particularly S is infinite

d.c.t. $\longrightarrow \overset{?}{=} \sum_{z\in S} \lim_{n\to\infty} \cdots$

$$= \left(\sum_{z\in S} \pi(z)\right)\frac{1}{m_x}$$

$$= 1\cdot \frac{1}{m_x}$$

$$= \frac{1}{m_x}, \qquad \forall x\in S.$$

Step 2. to show "$\exists$": namely, to show $\left(\frac{1}{m_x}\right)_{x\in S}$ is a SD.

$\underbrace{m_x}_{\in(0,1)}$

i.e.

(i) $\sum_{x\in S} \frac{1}{m_x} = 1$

(ii) $\sum_{x\in S} \frac{1}{m_x} P(x,y) = \frac{1}{m_y}$ (Stationary)

step 2.1. "$\le$" true in (i) & (ii).

claim: $\sum_{x\in S} \frac{1}{m_x} \le 1$

proof: P^m: Markov matrix

$$\sum_{x\in S} P^m(z,x) = 1, \qquad \forall z\in S$$

$$\frac{\sum_{m=1}^{n}(\dots)}{n} \Rightarrow 1 = \sum_{x\in S} \frac{\sum_{m=1}^{n} P^m(z,x)}{n}$$

$$\Rightarrow 1 = \lim_{n\to\infty} \sum_{x\in S} (\dots)$$

$$= \lim_{n\to\infty} \sum_{x\in S} (\dots)$$

$$\geq \sum_{x\in S} \lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(z,x)}{n}$$

$$= \sum_{x\in S} \frac{1}{m_x}$$

Claim: $\sum_{x\in S} \frac{1}{m_x} P(x,y) \leq \frac{1}{m_y}, \quad \forall y \in S$

Proof:

$$\underbrace{(\underline{\underline{P^m}} \cdot \underline{\underline{P}})(z,y)}_{= \sum_{x\in S} P^m(z,x)P(x,y)} = P^{m+1}(z,y) \quad \forall z, \forall y$$

$$\frac{\sum_{m=1}^{n}(\dots)}{n} \Rightarrow \underset{\underset{(\#S \leq \infty)}{=}}{\sum_{x\in S}} \frac{\sum_{m=1}^{n} P^m(z,x)}{n} P(x,y) = \frac{\sum_{m=1}^{n} P^{m+1}(z,y)}{n} \quad [2, n+1]$$

$$= \frac{\sum_{m=2}^{n+1} P^m(z,y)}{n}$$

$$= \underbrace{\frac{\sum_{m=1}^{n+1} P^m(z,y)}{n+1}}_{\xrightarrow[n\to\infty]{} \frac{1}{m_y}} \cdot \underbrace{\frac{n+1}{n}}_{\left(\xrightarrow[n\to\infty]{} 1\right)} - \underbrace{\frac{P(z,y)}{n}}_{\xrightarrow[n\to\infty]{} 0}$$

$\underline{n\to\infty}$:

$$\sum_{x\in S} \underbrace{\lim_{n\to\infty}(\dots)}_{= \lim_{n\to\infty}(\dots) = \frac{1}{m_x}P(x,y)} \leq \lim_{n\to\infty} \sum_{x\in S}(\dots) = \lim_{n\to\infty} \sum_{x\in S} \underbrace{(\dots)}_{\geq 0} = \frac{1}{m_y}$$

$$\therefore \quad \sum_{x\in S} \frac{1}{m_x} P(x,y) \le \frac{1}{m_y}, \quad \forall y \in S.$$

<u>Step 2.2</u>. "=" true in (i) & (ii)

<u>Claim</u>: $$\sum_{x\in S} \frac{1}{m_x} P(x,y) \overset{(\le)}{=} \frac{1}{m_y}, \quad \forall y \in S.$$

("=" in (ii))

<u>Proof</u>: otherwise, $\exists\, y_0 \in S$, s.t.

$$\sum_{x\in S} \frac{1}{m_x} P(x,y_0) < \frac{1}{m_{y_0}}.$$

$$1 \overset{\text{Step 2.1}}{\geq} \sum_{y\in S} \frac{1}{m_y} > \underbrace{\sum_{y\in S} \left(\sum_{x\in S} \frac{1}{m_x} P(x,y) \right)}$$

$$= \sum_{x\in S} \sum_{y\in S} \left(\frac{1}{m_x} P(x,y) \right)$$

$$= \sum_{x\in S} \left\{ \frac{1}{m_x} \underbrace{\left[\sum_{y\in S} P(x,y) \right]}_{=1} \right\}$$

$$= \sum_{x\in S} \frac{1}{m_x} \geq 0$$

Contradiction: "$A < A$ for a finite $A \in [0,1]$"

#

$(\leq)$

Claim: $\sum_{x\in S} \frac{1}{m_x} = 1$,

Pf. We know: $\underbrace{\sum_{x\in S} \frac{1}{\underbrace{m_x}_{\in(0,1)}} \leq 1}_{\text{convergent}}$

then set

$$(0,1] \ni \underbrace{c}_{?=1} = \sum_{x\in S} \frac{1}{m_x}$$

$$\xRightarrow{\frac{1}{c}} \quad 1 = \sum_{x\in S} \frac{1}{c m_x}$$

then

$$\left(\frac{1}{\underbrace{c m_x}_{>0}}\right)_{x\in S} \quad \text{is a SD.}$$

① prob.
② stationary

But, SD is unique (By step 1)

$\therefore \frac{1}{c m_x} = \frac{1}{m_x}$, $\forall x$

i.e. $c = 1$.

$\therefore \sum_{x\in S} \frac{1}{m_x} = 1$. #

Remarks :

① Heuristic understanding in case S is finite

$$\lim_{n\to\infty} P^n(x,y) = \pi(y), \quad \forall x, y \Rightarrow \pi \text{ is a SD}$$

$\uparrow$ prob. vector

$\sum_{y\in S} P^n(x,y) = 1 \quad \therefore \sum_{y\in S} \pi(y) = 1.$

$$\Downarrow \quad \not\Uparrow$$

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(x,y)}{n} = \lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = \underbrace{\pi(y)}_{S.D.}, \quad \forall x, y \in S$$

$\overset{\text{showed}}{=} \frac{1}{m_y}$

(irreducible recurent)

② Thm tells us a way to compute

$$m_x = E_x(T_x)$$

$\uparrow$ mean (average) returning time

by looking for the SD in case the chain is irreducible & (+)-recurrent :

$$(0,\infty) \ni E_x(T_x) = \frac{1}{\pi(x) \in (0,1)}, \quad \forall x \in S$$

π : SD (unique)

Coro#1: An irreducible MC having finite states admits a unique SD.

pf: Chain must be (+)-recurrent from the previous lecture. #

Coro #2. Let MC be irreducible. Then it has a SD iff it is (+)-recurrent.

$\Leftarrow$ By thm

$\Rightarrow$ need a proof

Proof. it suffices to show "$\Rightarrow$":

Otherwise, no (+)-recurrent, then all states could be either 0-recurrent or transient.

In both cases,

$$\lim_{n\to\infty} \sum_{m=1}^{n} \frac{P^m(x,y)}{n} = 0 \qquad \forall x, y \in S$$

Assume: π is a SD for the chain.

$\forall x \in S$, (P^m: Markov, $\forall m \geq 1$)

$$\pi(x) = \sum_{z\in S} \pi(z) P^m(z,x)$$

(stationary)

$$\frac{\sum_{m=1}^{n}(\dots)}{n} \Rightarrow \pi(x) = \sum_{z\in S} \pi(z) \boxed{\frac{\sum_{m=1}^{n} P^m(z,x)}{n}} \xrightarrow[n\to\infty]{} 0 \quad \forall z, x \in S$$

$$\xRightarrow{n\to\infty} \pi(x) = \lim_{n\to\infty} \sum_{z\in S} \{\cdots\}$$

$$\overset{\text{B.C.T.}}{=} \sum_{z \in S} \lim_{n \to \infty} \{ \cdots \}$$

$$= \sum_{z \in S} \pi(z) \cdot 0$$

$$= \sum_{z \in S} 0$$

$$= 0, \qquad \forall x \in S$$

π can NOT be a prob. vector

Contradiction to "π is SD"

#

e.g. Recall

$$P = \begin{bmatrix} q & p & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$p > 0$
$q > 0$
$p + q = 1$

is irreducible BD chain

Fact:

- Recurrent $\iff q \geq p$
- $\exists$ SD $\iff q > p$

then,

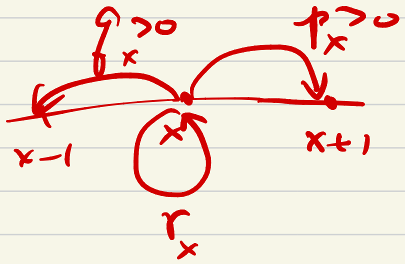
$$(+)\text{-recurrent} \overset{\text{Coro}}{\iff} \exists \text{ SD} \iff q > p.$$

$$q = p : \begin{cases} \text{No SD} \\ \text{all states are 0-recurrent} \end{cases}$$

$$q < p : \begin{cases} \text{No SD} \\ \text{all states are transient.} \end{cases}$$

Exercise :

For a general BD chain (irreducible)



find conditions to determine if

a state is $\begin{cases} (+)\text{-recurrent} \\ 0\text{-recurrent} \\ \text{transient.} \end{cases}$

Remark :

$$S = \underset{(\text{disjoint})}{S_T \cup} \Big(\bigcup_{i=1}^{k} C_i \Big) \qquad k \leq \infty$$

irreducible closed set of recurrent states

Let $C = C_i$ for some i.

Assume : $C = C_i$ (+)-recurrent

Then, MC restricted to C has a unique SD:

$$\pi_C(x) = \frac{1}{m_x}, \quad \forall x \in C$$

$m_x = E_x(T_x) \in (0, \infty)$.

Def.

$$\pi(x) = \begin{cases} \pi_C(x) = \frac{1}{m_x} & x \in C \\ 0 & \text{otherwise} \end{cases}$$

then π is a SD of MC over the whole state space S, called the SD concentrated on $\underline{\underline{C}}$

irreducible & closed set of (+)-recurrent states.

Notes:

① For instance

$C_1 \rightsquigarrow \pi_1$: SD concentrated on C_1

$C_2 \rightsquigarrow \pi_2$: SD ———————— C_2

$$\pi \overset{\text{def.}}{=} \lambda \pi_1 + (1-\lambda)\pi_2$$

$$0 \leq \lambda \leq 1.$$

is a SD of MC over S. #

② If no C_i in $\bigcup_{i=1}^{k} C_i$ is (+)-recurrent

(all states are null-recurrent)

Then this chain has no SD. #