March 10th :

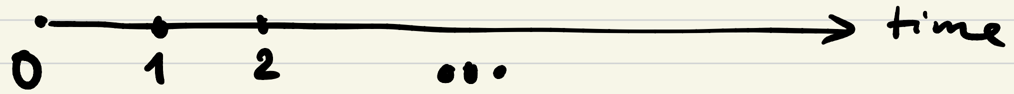
§2. Average number of visits to a state

Setting : MC $\{X_n\}_{n=0}^{\infty}$,

S : state space, $\#S \leqq \infty$

P : Markov transition function

Recall: $y \in S$



r.v. $N(y) = \sum_{m=1}^{\infty} 1_y (X_m)$

total no of visits to y

for all positive time

r.v. : For $n \geq 1$,

$$N_n(y) \stackrel{\text{def.}}{=} \sum_{m=1}^{n} 1_y (X_m)$$

this is the no of visits to $y \in S$

in n-steps (i.e. at positive times

1 to n)

Recall:

$$E_x(N(y)) = \sum_{m=1}^{\infty} P^m(x,y)$$

So, similary,

$$E_x(N_n(y)) = \sum_{m=1}^{n} P^m(x,y)$$

We want to study:

$\dfrac{N_n(y)}{n}$: proportion / visit frequency

(i.e. no of visits to y / unit time

$E_x\left(\dfrac{N_n(y)}{n}\right)$: (average) visit frequency

long-run behavior

$n \longrightarrow \infty$

Remark: If y is transient, then answer is easy:

$(\Leftrightarrow P_x(N(y)=\infty)=0)$

$$\lim_{n\to\infty} N_n(y) \overset{\text{Prob.}}{=} N(y) \ (<\infty)$$

(Ref.: $\xi_n \xrightarrow{P} \xi$ as $n \to \infty$

$\Leftrightarrow \lim_{n\to\infty} P(|\xi_n - \xi| > \epsilon) = 0, \ \forall \epsilon > 0)$

$$\Rightarrow \lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{Prob.}}{=} 0$$

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = E_x\left(\lim_{n\to\infty} \frac{N_n(y)}{n}\right) = 0$$

this means:

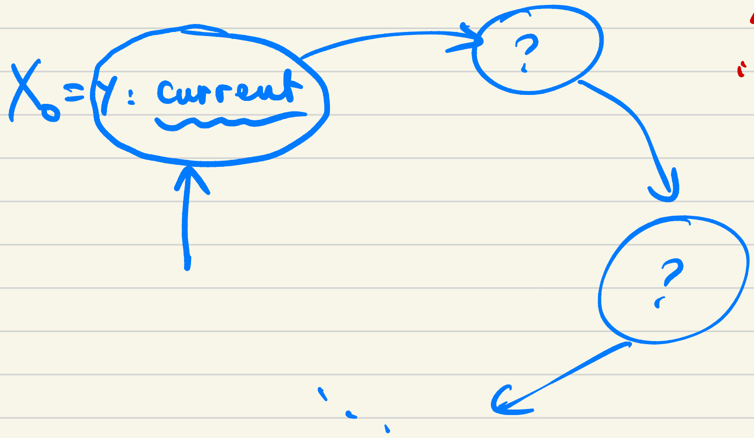
the visit frequency to any transient state must be zero in the long-run.

Question: What if y recurrent ?

$\rho_{yy} = 1$

i.e.

$P_y(T_y < \infty) = 1$



Expect:

" visit frequency $=$ $\dfrac{1}{\text{returning time}}$ "

i.e. no of visits to y / unit time

i.e. the waiting time for the chain from y to return back to y.

Thm: Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible recurrent MC, then

$$\lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{prob.}}{=} \frac{1}{m_y}$$

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = \frac{1}{m_y}, \quad \forall x \in S$$

where $m_y \overset{\text{def.}}{=} E_y(T_y)$

a recurrent state

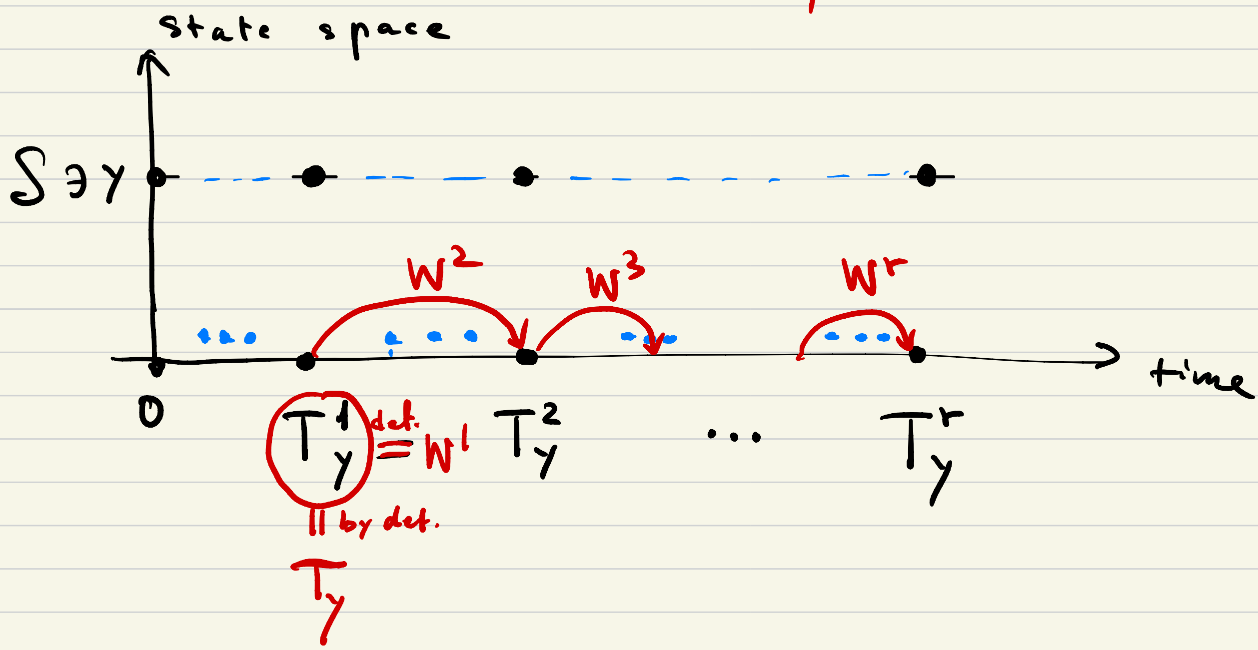
means the average time of returning to y for the chain starting from y.

Proof: $X_0 = y$ (current).

For $r = 1, 2, \ldots$

$$T_y^r \stackrel{\text{def.}}{=} \min\{ n \geq 1 : N_n(y) = r \}$$

r.v. denoting the min positive n for the r^{th} visits to y



$$\begin{cases} W^1 \stackrel{\text{def.}}{=} T_y^1 = \min\{ n \geq 1, N_n(y) = 1 \} = T_y \\ \qquad\qquad\qquad\qquad (= \min\{ n \geq 1 : X_n = y \}) \\ W^r \stackrel{\text{def.}}{=} T_y^r - T_y^{r-1}, \quad r = 1, 2, \ldots \end{cases}$$

$\Rightarrow$ a sequence of r.v.s

$$W^1, \; W^2, \; \ldots, \; W^r, \; \ldots$$

$W^1 = T_y$

is i.i.d.

note: $T_y^r = W^1 + W^2 + \cdots + W^r$

Strong law of large number $\Longrightarrow$

$$\lim_{n\to\infty} \frac{T_y^r}{n} = \lim_{n\to\infty} \frac{W^1 + W^2 + \cdots + W^r}{n} \overset{\text{prob.}}{=\!=} E_y(W^1)$$

$$= E_y(T_y)$$

$$= m_y$$

Note:

Given $\underset{\sim\sim\sim}{N_n(y) = r}$,

(i.e. r visits to y in n-steps)

(otherwise, $T_y^{r+1} \le n$, then $N_n(y) \geq r+1$)

then $T_y^r \le n < T_y^{r+1}$

$\therefore$

$$m_y \xleftarrow[r\to\infty]{\text{prob.}} \frac{T_y^r}{r} \le \frac{n}{N_n(y)} = \frac{n}{r} < \boxed{\frac{T_y^{r+1}}{r}}$$

consider the event $N_n(y) = r$

$$= \frac{T_y^{r+1}}{r+1} \cdot \frac{r+1}{r}$$

$\frac{T_y^{r+1}}{r+1} \xrightarrow[r\to\infty]{\text{prob.}} m_y$, $\quad \frac{r+1}{r} \to 1$

Squeezing law $\Rightarrow$

$$\lim_{n\to\infty} \frac{n}{N_n(y)} \overset{\text{prob.}}{=} m_y \quad . \quad \#$$

Moreover,

due to dominated convergence thm

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) \overset{?}{=} E_x\left(\lim_{n\to\infty} \frac{N_n(y)}{n}\right)$$

$\boxed{\text{YES}}$

$$= E_x\left(\frac{1}{m_y}\right)$$

$$= \frac{1}{m_y} \quad . \qquad \#$$

FYI:

D.C.T.:

$$\begin{cases} * \quad \xi_n(\omega) \xrightarrow[n\to\infty]{} \xi(\omega) \qquad \forall \omega \in \Omega \\ * \quad \exists\, \eta \ (\text{r.v.}) \ \text{ s.t. } \\ \qquad |\xi_n| \le \eta, \quad E(\eta) < \infty \end{cases}$$

$$\Rightarrow \quad \lim_{n\to\infty} E(|\xi_n - \xi|) = 0$$

particularly

$$\lim_{n\to\infty} E(\xi_n) = E(\xi)$$

$$\left(= E\left(\lim_{n\to\infty} \xi_n\right)\right)$$

$$\#$$

Final remark: General situation

$$MC, \qquad S = \underset{\neq \phi}{S_R} \cup S_T = \left(\bigcup_{i=1}^{k} C_i\right) \cup S_T$$

$y \in S_R$

$$\lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{prob.}}{=} \frac{1_{\{T_y < \infty\}}}{m_y}$$

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = \frac{\rho_{xy}}{m_y}, \qquad \forall x \in S$$

$$\|$$

$$E_x\left(\frac{\mathbb{1}_{\{T_y<\infty\}}}{m_y}\right) = \frac{E_x(\mathbb{1}_{\{T_y<\infty\}})}{m_y}$$

$$= \frac{1 \cdot P_x(T_y<\infty)}{m_y}$$

$$= \frac{\rho_{xy}}{m_y}$$