# Math 4240 : Stochastic Processes

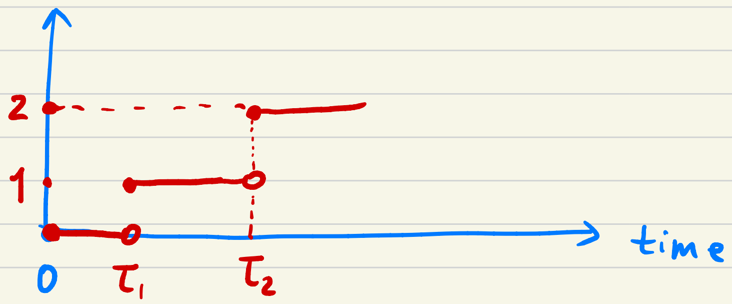
What's a stochastic Process ?

e.g. 1. check the weather in HK at

days 0, 1, 2, ...

| days | 0 | 1 | 2 | ... |
|---|---|---|---|---|
| weather | sunny | cloudy | rainy | ... |

e.g. 2. Count the no of arrivals




Sum :

$X_t$ : a "random variable" to denote the "state" of the "system" that is "changing in time $t$"

* If time is discrete,

$X_t$ , $t = 0, 1, 2, \ldots$

i.e. $X_0, X_1, X_2, \ldots$

discrete-in-time SP

* If time is continuous,

$X_t$ , $t \geq 0$

Continuous-in-time SP

Main subject :

① how to characterize $\{X_t\}_{t\geq 0}$

② how to find the long-run behaviour

Review on probability :

① Probability space :

$$(\Omega, \mathcal{F}, P)$$

$\Omega$ ↑ sample space : an element called the outcome

$\mathcal{F}$ ↑ event space = a collection of subsets of $\Omega$ (satisfies the properties of $\sigma$-algebra)

the element called the event

$P$ ↑ prob. measure : a function on $\mathcal{F}$ valued in $[0,1]$ :

① $P(\Omega) = 1$

② $0 \leq P(A) \leq 1$

③ $P(\underbrace{\cup A_i}_{\text{disjoint Union}}) = \sum_i P(A_i)$

e.g. toss 3 coins

1st 2nd 3rd

$$\Omega = \{ (H,H,H), (H,H,T), (H,T,H), (T,H,H), (H,T,T), (T,H,T), (T,T,H), (T,T,T) \}$$

↑ 8 outcomes : $\#\Omega = 8$

$\exists A \subset \Omega$ : $A \overset{\text{def.}}{=}$ "one and only one head occur"

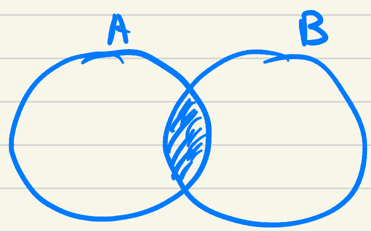
$\uparrow$ event $= \{(H,T,T), (T,H,T), (T,T,H)\}$

we like to consider

( $\sigma$- algebra )

## Conditional prob. :

- $P(B|A)$ = prob that B occurs given that A occurs.
- $P(B|A) \overset{\text{def.}}{=} \dfrac{P(A \cap B)}{P(A)}$ , if $P(A) \neq 0$




- A, B are independent if

$$P(B|A) = P(B)$$

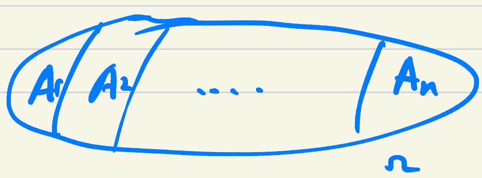
( or $P(A|B) = P(A)$

or $P(A \cap B) = P(A)\,P(B)$ )

- $P_A(\cdot) = P(\cdot \mid A)$

conditional prob measure

**Thm** If




$\Omega = \bigcup_{i=1}^{n} A_i$ (disjoint union) then, $\forall B$

① $$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

② $$\underbrace{P(A_i|B)}_{\text{prob that}} = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

prob that the cause $A_i$ occurs given that the consequence $B$ occurs.

Bayes' formula

## #2 : Random varible

given $(\Omega, \mathcal{F}, P)$ :

$$X : \Omega \to \mathbb{R}$$

$$\omega \mapsto X_\omega \in \mathbb{R}$$ $\mathcal{F}$- measurable function

- $X$ is a discrete r.v. if the image $X(\Omega)$ is discrete
- $X$ is a continuous r.v. if $X(\Omega)$ is $\mathbb{R}$ or some subinterval of $\mathbb{R}$.

Discrete r.v. :

$$X(\Omega) = S = \{k\}_{k=0}^{N} = \{0, 1, 2, \dots, N\}$$

$N \to$ finite or infinite

$S$: state space

0 = "rainy"
1 = "sunny"
2 = "cloudy"

Prob. density function :

| States $k$ | 0 | 1 | 2 | $\cdots$ | $k$ | $\cdots$ | $N$ |
|---|---|---|---|---|---|---|---|
| Prob that $X=k$ | $P(X=0)$ | $P(X=1)$ | $P(X=2)$ | $\cdots$ | $P(X=k)$ | $\cdots$ | $P(X=N)$ |
| | $\parallel$ $p_0$ | $\parallel$ $p_1$ | $\parallel$ $p_2$ | | $\parallel$ $p_k$ | | $\parallel$ $p_N$ |

$$(p_k)_{k=0}^{N} = \left(P(X=k)\right)_{k=0}^{N} : \text{ p.d.f. of } X$$

$$P(\underbrace{X=k}_{\substack{\text{an event} \\ \subset\Omega}}) = P\left(\left\{\omega\in\Omega : X_\omega = k\right\}\right)$$

examples:

① Binomial distribution:

$n$ independent trials

$p =$ success prob

$1-p =$ unsuccess prob.

$X \overset{\text{def.}}{=}$ no of successes in such $n$ trials

$X$ takes values in $\underbrace{\{0, 1, \cdots, n\}}_{\text{state space}}$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$k = 0, 1, \cdots, n$.

② Poisson distribution

$X \overset{\text{def.}}{=}$ no of arrivals in a unit time

Assume : $\lambda$ = arrival rate $> 0$

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$k = 0, 1, 2, \ldots$

③ Geometrix distri

$X \stackrel{\text{def.}}{=}$ no of trials you performed until the 1st success

| 1st | 2nd | 3rd | ... | (k-1)th | kth |
| --- | --- | --- | --- | --- | --- |
| F | F | F | ... | F | S |

$$P(X=k) = (1-p)^{k-1} p$$

$k = 1, 2, \ldots$

$$\left( \sum_{k=1}^{\infty} p_k = \left[ \sum_{k=1}^{\infty} (1-p)^{k-1} \right] p = \frac{1}{1-(1-p)} \cdot p = 1 \right)$$

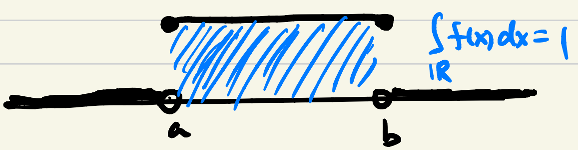
Continuous r.v. : $X$

$$P(\underbrace{a \le X \le b}_{\{\omega \in \Omega \,:\, a \le X_\omega \le b\}}) = \int_a^b f(x)\,dx$$

$f(x)$ : p.d.f. of $X$.

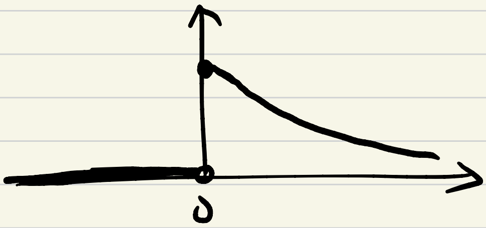
examples :

① uniform distribution




$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

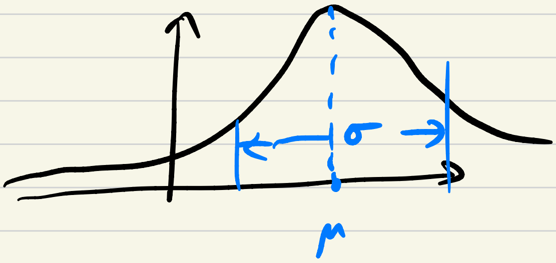
② exponential distri




$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{\mathbb{R}} f(x)\,dx = 1$$

③ Normal distribution




$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \underline{N(\mu, \sigma^2)}$$

## #3. Expectation & variance

$(\Omega, \mathcal{F}, P)$, $\quad X \atop S$ : discrete $(p_k)_{k\in S}$ or continuous $f(x)$

$$(p_k)_{k\in S} = P(X=k)_{k \in S}$$

$$\mu = E(X) = \sum_{k\in S} k\, p_k \quad \text{or} \quad \int_{\mathbb{R}} x f(x)\,dx$$

$\mu$: expectation or mean

$\sum_{k\in S} k\, p_k$: first moment

$$E(X^2) = \sum_{k\in S} k^2 p_k \quad \text{or} \quad \int_{\mathbb{R}} x^2 f(x)\,dx$$

$\sum_{k\in S} k^2 p_k$: 2nd moment

$$\sigma^2 = Var(X) = E(X-\mu)^2 = E(X^2) - \mu^2$$

$\uparrow$
variance of $X$ : to measure how spread $X$ is

Conditional expectation : $X, \quad S_X = \{i\}$
$Y, \quad S_Y = \{j\}$

discrete case :

$$p_{ij} = P(X=i, Y=j), \quad i \in S_X, \; j \in S_Y$$

joint distribution

$$E(Y \mid X=i) = \sum_{j \in S_Y} j \, P(Y=j \mid X=i)$$

$$= \sum_{j \in S_Y} j \frac{P(X=i, Y=j)}{P(X=i)} \qquad P(X=i) = \sum_{j \in S_Y} p_{ij}$$

$$\{X=i\} = \bigcup_{j \in S_Y} \{X=i, Y=j\}$$

disjoint

$$= \sum_{j \in S_Y} j \frac{p_{ij}}{\sum_{j \in S_Y} p_{ij}}$$

Continuous case :

$$P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v) \, dv \, du$$

joint distribution

$$E(Y \mid X=x) = \int_{R(Y)} y \frac{f(x,y)}{\boxed{f_X(x)}} dy$$

p.d.f. of $X$

$$f_X(x) = \int_{R(Y)} f(x,y) \, dy$$

— End of Review —