Lecture Six

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1 The finite interval

The wave equation with fixed ends:

$$\begin{cases} v_{tt} = c^2 v_{xx} & 0 < x < l, \\ v(x,0) = \phi(x), v_t(x,0) = \psi(x), \\ v(0,t) = v(l,t) = 0. \end{cases}$$

We extend them to the whole line

$$\phi_{ext} = \begin{cases} \phi(x) & 0 < x < l \\ -\phi(-x) & -l < x < 0 \\ extended\ to\ be\ of\ period\ 2l. \end{cases}$$

and

$$\psi_{ext} = \begin{cases} \psi(x) & 0 < x < l \\ -\psi(-x) & -l < x < 0 \\ extended\ to\ be\ of\ period\ 2l. \end{cases}$$

So the formula of v is

$$v(x,t) = \frac{1}{2}[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)ds.$$

Case (0,0), $x - ct \geq 0$, $x + ct \leq l$ and $t > 0$:

$$v(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Case (0,1), $x - ct \geq 0$, $x + ct \geq l$ and $x \leq l$:

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi(x-ct) - \phi(2l-x-ct)] + \frac{1}{2c}\int_{x-ct}^{l} \psi(s)ds - \frac{1}{2c}\int_{l}^{2l} \psi(2l-s)ds \\ &= \frac{1}{2}[\phi(x-ct) - \phi(2l-x-ct)] + \frac{1}{2c}\int_{x-ct}^{l} \psi(s)ds + \frac{1}{2c}\int_{l}^{2l-x-ct} \psi(s)ds \\ &= \frac{1}{2}[\phi(x-ct) - \phi(2l-x-ct)] + \frac{1}{2c}\int_{x-ct}^{2l-x-ct} \psi(s)ds. \end{aligned}$$

Case (1,1), $-l \le x - ct \le 0$ and $2l \ge x + ct \ge l$:

$$\begin{aligned} v(x,t) &= \frac{1}{2}[-\phi(ct-x) - \phi(2l-x-ct)] + \frac{1}{2c}[\int_{x-ct}^{0} -\psi(-s)ds + \int_0^l \psi(s)ds - \int_l^{x+ct} \psi(2l-s)ds] \\ &= \frac{1}{2}[-\phi(ct-x) - \phi(2l-x-ct)] + \frac{1}{2c}\int_{ct-x}^{2l-x-ct} \psi(s)ds. \end{aligned}$$

Case (1,2), $-l \le x - ct \le 0$, $x + ct \ge 2l$ and $x \le l$:

$$\begin{aligned} v(x,t) &= \frac{1}{2}[-\phi(ct-x) + \phi(x+ct-2l)] \\ &= +\frac{1}{2c}[\int_{x-ct}^{0} -\psi(-s)ds + \int_0^l \psi(s)ds - \int_l^{2l} \psi(2l-s)ds + \int_{2l}^{x+ct} \psi(s-2l)ds] \\ &= \frac{1}{2}[-\phi(ct-x) + \phi(x+ct-2l)] - \frac{1}{2c}\int_{x+ct-2l}^{ct-x} \psi(s)ds, \end{aligned}$$

which depends only on the initial values on the interval $[x+ct-2l, ct-x]$.

Similarly, you can derive the formula of v on each domain of Chapter 3 Figure 6 in the textbook.

2 Waves with a source

We are going to solve wave equations with a source term f on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{1}$$

where $f(x,t)$ is a given function. For instance, $f(x,t)$ could be interpreted as an external force acting on an infinitely long vibrating string. This is an inhomogeneous linear equation. If you can find a solution u_f to the equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & -\infty < x < \infty \\ u(x,0) = 0 \\ u_t(x,0) = 0, \end{cases}$$

then with the solution $u_{hom} = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi$ for the homogeneous linear wave equation with the initial dates ϕ and ψ, you will get the solution $u = u_f + u_{hom}$ for the problem (1).

We will show that

$$u_f(x,t) = \frac{1}{2c}\int\int_{\triangle} f = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds, \tag{2}$$

where $\triangle$ is the domain of dependence of (x,t) (characteristic triangle).

The formula illustrates the effect of a force f on $u(x,t)$ is obtained by simply integrating f over the past history of the point (x,t) back to the initial time $t=0$. This is another example of the causality principle.

This is a well-posed problem.

- *Existence* is from the explicit formula.
- The *uniqueness* can be deduced from the Energy identity see Lec 5, section 2.
- The *stability* is from: suppose u_1 is the solution with data (ϕ_1, ψ_1, f_1) and u_2 is the solution with data (ϕ_2, ψ_2, f_2). Then the difference $u = u_1 - u_2$ is given by the following formula due to linearity:

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi_1(x+ct) - \phi_2(x+ct) + \phi_1(x-ct) - \phi_2(x-ct)] \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct}(\psi_1 - \psi_2) \\ &+ \frac{1}{2c}\int\int_{\triangle}(f_1 - f_2). \end{aligned} \tag{3}$$

Define the uniform norm

$$\|w\| = \max_{-\infty<x<\infty}|w(x)|$$

and

$$\|w\|_T = \max_{-\infty<x<\infty, 0\le t\le T}|w(x,t)|$$

where T is fixed.

So from the formula (1), we have the estimate

$$\|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\|\psi_1 - \psi_2\| + \frac{T^2}{2}\|f_1 - f_2\|_T.$$

If $\|\phi_1 - \phi_2\|$, $\|\psi_1 - \psi_2\|$ and $\|f_1 - f_2\|_T$ are small, then $\|u_1 - u_2\|_T$ is also small. This proved the *stability*.

Now we start to prove the formula (2).

Proof. Method of Characteristic Coordinates. We intoduce coordinates $\xi = \frac{x+ct}{2c}$, $\eta = \frac{ct-x}{2c}$ such that

$$u_{tt} - c^2 u_{xx} = u_{\xi\eta} = f(c\xi - c\eta, \xi + \eta).$$

Integrate along η then along ξ

$$u = \int^{\xi}\int^{\eta} f(c\xi' - c\eta', \xi' + \eta')d\eta' d\xi',$$

where the lower limits of integration are arbitrary. We may make a particular choice of the lower limits to find a particular solution in the domain $\triangle' = \{(\xi', \eta') | \xi' + \eta' \geq 0, \xi' \leq \xi, \eta' \leq \eta\}$

$$\begin{aligned} u(x,t) &= \int_{\eta}^{\xi} \int_{-\xi'}^{\eta} f(c\xi' - c\eta', \xi' + \eta') d\eta' d\xi' \\ ? &= \frac{1}{2c} \int \int_{\triangle} f(x,t) dx dt. \end{aligned}$$

Try to figure out the last equality by yourself. $\square$