Lecture Four

January 20, 2021

We begin to solve one dimensional wave equations.

1 Wave equation on the whole line

Find the general solutions of the equation for $c \neq 0$

$$u_{tt} - c^2 u_{xx} = 0 \quad for \quad -\infty < x < \infty.$$

Proof. Method one: We rewrite the equation into the form

$$u_{tt} - c^2 u_{xx} = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0. \quad (1)$$

Let

$$v = u_t + cu_x. \quad (2)$$

From the equation (1), v satisfies a transport equation

$$v_t - cv_x = 0.$$

So we solve v from this equation

$$v(x,t) = h(x + ct),$$

where h is any function.
Then we try to solve u from the equation (2)

$$u_t + cu_x = h(x + ct). \quad (3)$$

First we solve homogenous linear equation

$$\bar{u}_t + c\bar{u}_x = 0.$$

So the general solution of this equation is

$$\bar{u}(x,t) = g(x - ct).$$

If we find a particular solution of the inhomogenous linear equation (3) $u(x,t) = f(x+ct)$, where $f'(s) = \frac{h(s)}{2c}$.

Then the general solution of the equation (3) is

$$u(x,t) = f(x+ct) + g(x-ct).$$

This is because of linearity suppose u^1, u^2 are the solutions then $u^1 - u^2$ is the solution in the form of homogenous linear equation.

Method two: Find a characteristic coordinates (ξ, η) such that

$$\begin{aligned} \frac{\partial t}{\partial \xi} &= 1 \\ \frac{\partial x}{\partial \xi} &= c \\ \frac{\partial t}{\partial \eta} &= 1 \\ \frac{\partial x}{\partial \eta} &= -c. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \\ \frac{\partial}{\partial \eta} &= \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}. \end{aligned}$$

Now we get

$$\begin{aligned} t &= \xi + \eta \\ x &= c\xi - c\eta. \end{aligned}$$

If we denote $\tilde{u}(\eta, \xi) = u(x,t)$, by the chain rule

$$\begin{aligned} \tilde{u}_\xi &= \frac{\partial t}{\partial \xi}u_t + \frac{\partial x}{\partial \xi}u_x \\ &= u_t + cu_x. \end{aligned}$$

And

$$\begin{aligned} \widetilde{u}_{\xi\eta} &= \frac{\partial t}{\partial \eta}u_{tt} + \frac{\partial x}{\partial \eta}u_{tx} + c\frac{\partial t}{\partial \eta}u_{xt} + c\frac{\partial x}{\partial \eta}u_{xx} \\ &= u_{tt} - cu_{tx} + cu_{xt} - c^2u_{xx} = 0. \end{aligned}$$

So we get

$$\tilde{u}(\xi, \eta) = \tilde{f}(\xi) + \tilde{g}(\eta).$$

We rewrite it back to the orginal system

$$\begin{aligned} u(x,t) &= \tilde{f}(\frac{ct+x}{2c}) + \tilde{g}(\frac{ct-x}{2c}) \\ &= f(ct+x) + g(x-ct), \end{aligned}$$

where f and g are any functions. □

2 Initial value problem

Find the solution for the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x), \end{cases}$$

where ϕ and ψ are given functions.

Proof. The general form of solutions to the wave equation on the whole line is

$$u(x,y) = f(ct+x) + g(ct-x). \tag{4}$$

Setting $t = 0$, we have

$$\begin{aligned} u(x,0) = \phi(x) &= f(x) + g(-x) \\ u_t(x,0) = \psi(x) &= cf'(x) + cg'(-x). \end{aligned} \tag{5}$$

Differentiating the first equation to get

$$\phi'(x) = f'(x) - g'(-x).$$

We can solve $f'$and g' from the above equations

$$\begin{aligned} f'(x) &= \frac{c\phi'(x) + \psi(x)}{2c} \\ g'(-x) &= \frac{\psi(x) - c\phi'(x)}{2c}. \end{aligned}$$

We integrate once to get f, g

$$\begin{aligned} f(x) &= \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(x')dx' + c_1 \\ -g(-x) &= -\frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(x')dx' + c_2, \end{aligned}$$

where c_1, c_2 are constants.

Because of equation (5), we have $c_1 = c_2$. So the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\phi(ct+x) + \frac{1}{2c}\int_0^{ct+x} \psi(x')dx' + \frac{1}{2}\phi(-ct+x) - \frac{1}{2c}\int_0^{-ct+x} \psi(x')dx' \\ &= \frac{1}{2}[\phi(ct+x) + \phi(-ct+x)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'. \end{aligned}$$

This formula is due to d'Alembert which is called d'Alembert formula. □