Lecture 19

March 25, 2021

Definition 1. We say that an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ pointwise in (a, b) if it converges to $f(x)$ for each $a < x < b$. That is , for each $a < x < b$ we have

$$|f(x) - \sum_{n=1}^{N} f_n(x)| \to 0 \quad as\ N \to \infty.$$

Theorem 2. *(Pointwise Convergence of Classical Fourier Series)*

i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b) provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$.

The sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}[f(x+) + f(x-)]$$

for all $a < x < b$.

The sum is $\frac{1}{2}[f_{ext}(x+) + f_{ext}(x-)]$ for all $-\infty < x < \infty$, where $f_{ext}(x)$ is the extended function (periodic, odd periodic or even periodic).

Definition 3. We say that the series converges uniformly to $f(x)$ in $[a, b]$ if

$$\max_{a \le x \le b} |f(x) - \sum_{n=1}^{N} f_n(x)| \to 0,$$

as $N \to \infty$.

Theorem 4. *(Uniform Convergence) The classical Fourier series (full, sine, and cosine) converges to $f(x)$ uniformly on $[a, b]$ provided that*

i) $f(x)$,$f'(x)$ exist and are continuous for $a \le x \le b$ and

ii) f(x) satisfies the given boundary conditions.

Example 5. The Fourier sine series of the function $f(x) \equiv 1$ on the interval $(0, \pi)$ is

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx. \tag{1}$$

Although it converges at each point, this series does not converge uniformly on $[0, \pi]$. One reason is that the series equals zero at both endpoints (0 and π) but the function is 1 there. Condition (ii) of Theorem 4 is not satisfied.

The Fourier series (1) can not be differentiated term by term.

Definition 6. We say the series converges in the mean-square (or L^2) sense to $f(x)$ in (a, b) if

$$E_N = \int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \to 0$$

as $N \to \infty$.

Theorem 7. *(L^2 Convergence) The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which*

$$\int_a^b |f(x)|^2 dx$$

is finite.

Theorem 8. *The Fourier series of $f(x)$ converges to $f(x)$ in the mean-square sense if and only if*

$$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx. \tag{2}$$

Proof. Mean-square convergence means that the remainder

$$E_N = \|f\|^2 - \sum_{n \le N} |A_n|^2 \|X_n\|^2 \to 0,$$

which in turn means (2), known as Parseval's equality. □

Corollary 9. *If $\int_a^b |f(x)|^2 dx$ is finite, then the Parseval equality (2) is true.*

Example 10. Consider once again the Fourier series

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx.$$

Parseval's equality asserts that

$$\sum_{n \text{ odd}} (\frac{4}{n\pi})^2 \int_0^{\pi} \sin^2 nx dx = \int_0^{\pi} 1^2 dx.$$

That is

$$\sum_{n \text{ odd}} (\frac{4}{n\pi})^2 \frac{\pi}{2} = \pi.$$

In other words,

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$