Lecture 17

1 Inhomogeneous Boundary conditions

We are going to solve equations with inhomogeneous boundary conditions.

For example

$$\begin{cases} u_t = ku_{xx} & 0 < x < l \\ u(0,t) = h(t) \\ u(l,t) = j(t) \\ u(x,0) = 0. \end{cases}$$

We can not solve it by simply using the previous separation of variables method. Because suppose $u(x,t) = X(x)T(t)$.

Then from diffusion equation, we have

$$-\frac{X''(x)}{X(x)} = -\frac{T'(t)}{kT(t)} = \lambda.$$

If $\lambda = \beta^2 > 0$, then

$$X(x) = c\cos\beta x + d\sin\beta,$$

and

$$T(t) = be^{-\lambda kt}.$$

From the initial condition $u(x,0) = 0$, we have $X(x)T(0) = 0$. But suppose $T(0) = b \neq 0$, so $X(x) \equiv 0$. This gives $u = 0$ a trivial solution. If $b = 0$, then we also have a trivial solution $u = 0$.

If $\lambda = 0$ or $\lambda < 0$, we will also get a trivial solution. This does not coincides the boundary conditions if $h \neq 0$ or $j \neq 0$.

In conclusion, the separation of variables method does not work for this problem.

From the Fourier convergence theorems, if u and u' are continuous functions then we have for each fixed t

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{l}, \tag{1}$$

where the coefficients are given by

$$u_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\frac{n\pi x}{l}dx.$$

Here we do not insist that the series converge at the endpoints but only inside de interval. We can not differentiate the series (1) term by term.

By expanding functions u_t and u_{xx}, we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi x}{l} \qquad (2)$$

with coefficients

$$v_n(t) = \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi x}{l}dx,$$

and

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} w_n(t)\sin\frac{n\pi x}{l} \qquad (3)$$

with coefficients

$$w_n(t) = \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{l}dx.$$

By direct computation, we can represent v_n and w_n by u_n

$$v_n(t) = \frac{du_n(t)}{dt}$$

and

$$w_n(t) = -\frac{n^2\pi^2}{l^2}u_n(t) - \frac{2n\pi}{l^2}[j(t)(-1)^n - h(t)].$$

From (2), (3) and the diffusion equation, we have the equation for u_n,

$$\frac{du_n}{dt} = k\{-\frac{n^2\pi^2}{l^2}u_n(t) - 2n\pi l^{-2}[(-1)^n j(t) - h(t)]\}$$

with initial condition $u_n(0) = 0$.

The solution is

$$u_n(t) = -2nk\pi l^{-2}\int_0^t e^{-\frac{n^2\pi^2}{l^2}k(t-s)}[(-1)^n j(s) - h(s)]ds. \qquad (4)$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l},$$

with the coefficient given by (4).

Exercise 1. Using the above method to solve the inhomogeneous wave problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \qquad 0 < x < l \\ u(0,t) &= h(t) \\ u(l,t) &= k(t) \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x). \end{aligned}$$

2 Method of shifting the data

By subtraction, the boundary data can be made homogeneous. We can choose

$$\bar{u}(x,t) = (1 - \frac{x}{l})h(t) + \frac{x}{l}k(t).$$

And let $v(x,t) = u(x,t) - \bar{u}(x,t)$ which satisfies the homogeneous boundary problem

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f(x,t) - \bar{u}_{tt}(x,t) \\ v(0,t) &= 0 = v(l,t) \\ v(x,0) &= \phi(x) - \bar{u}(x,0) \\ v_t(x,0) &= \psi(x) - \bar{u}_t(x,0). \end{aligned}$$

After solve $v(x,t)$, we can get $u(x,t) = v(x,t) + \bar{u}(x,t)$.

In some cases, the boundary condition and the differential equation can simultaneously be made homogeneous by subtracting any known function that satisfies them.

Case 1. If h, k, and $f(x)$ are independent of time then we can first solve

$$\begin{aligned} -c^2 \bar{u}_{xx}(x) &= f(x) \\ \bar{u}(0) &= h \\ \bar{u}(l) &= k. \end{aligned}$$

Then let $v(x,t) = u(x,t) - \bar{u}(x)$ which satisfies

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \\ v(0,t) = v(l,t) &= 0 \\ v(x,0) &= \phi(x) - \bar{u}(x) \\ v_t(x,0) &= \psi(x). \end{aligned}$$

Case 2. If $f(x,t) = F(x)\cos wt$, $h(t) = H\cos wt$ and $k(t) = K\cos wt$. First we solve

$$\begin{aligned} -c^2 u_0''(x) - w^2 u_0(x) &= F(x) \\ u_0(0) &= H \\ u_0(l) &= K. \end{aligned}$$

Then let $v(x,t) = u(x,t) - u_0(x)\cos wt$ which satisfies

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \\ v(0,t) = v(l,t) &= 0 \\ v(x,0) &= \phi(x) - u_0(x) \\ v_t(x,0) &= \psi(x). \end{aligned}$$