Lecture 12

March 1, 2021

1 Fourier Series

In this lecture, we are going to find the coefficients in the Fourier series.

Let us begin with the *Fourier sine series* in the interval $(0, l)$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \tag{1}$$

We will try to find the coefficients A_n.

The key observation is the formula

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \quad if \quad m \neq n,$$

m and n being positive integers.

Proof. There is a trigonometric identity

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b).$$

The integral equals

$$\begin{aligned}
\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx &= \int_0^l \frac{1}{2}\cos(\frac{(n-m)\pi x}{l}) - \frac{1}{2}\cos(\frac{(n+m)\pi x}{l})dx \\
&= \frac{l}{2(n-m)\pi}\sin(\frac{(n-m)\pi x}{l})|_0^l \\
&\quad -\frac{l}{2(n+m)\pi}\sin(\frac{(n+m)\pi x}{l})|_0^l \\
m \neq n &= 0.
\end{aligned}$$

□

Suppose ϕ has the Fourier sine series in $(0,l)$ (1). Let's multiply (1) by $\sin(\frac{m\pi x}{l})$ and integrate in the integral

$$\begin{aligned}\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx &= \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx \\ &= A_m \int_0^l \sin^2\frac{m\pi x}{l}dx \\ &= A_m \int_0^l \frac{1}{2}-\frac{1}{2}\cos\frac{2m\pi x}{l}dx \\ &= \frac{l}{2}A_m.\end{aligned}$$

Therefore,

$$A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx. \qquad (2)$$

That is if ϕ has the Fourier sine series in $(0,l)$ (1), then the coefficients must be given by (2).

Example 1. Let $\phi(x)\equiv 1$ in the interval $(0,l)$. The A_m is

$$\begin{aligned}A_m &= \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}dx = -\frac{2l}{lm\pi}\cos\frac{m\pi x}{l}\Big|_0^l \\ &= \frac{2}{m\pi}(1-(-1)^m).\end{aligned}$$

So the Fourier sine series is

$$1 = \frac{4}{\pi}(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\cdots).$$

Exercise 2. Prove that the formula

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0 \quad if \quad m\neq n,$$

m and n being nonnegative integers.
And

$$\int_0^l \cos^2\frac{m\pi x}{l}dx = \frac{l}{2}.$$

Suppose that ϕ has the *Fourier cosine series* in $(0,l)$

$$\phi(x) = \frac{1}{2}A_0+\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{l},$$

then the coefficients A_n must be given by

$$A_0 = \frac{2}{l}\int_0^l \phi(x)dx,$$

and

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}dx.$$

Example 3. The function $\phi(x) \equiv 1$ has a Fourier cosine series with coefficients

$$A_0 = \frac{2}{l}\int_0^l dx = 2,$$

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \cos\frac{m\pi x}{l}dx \\ &= \frac{2}{m\pi}\sin\frac{m\pi x}{l}|_0^l \\ &= 0 \end{aligned}$$

for $m \neq 0$. So we have

$$1 = 1 + 0 + 0 + \cdots.$$

The full Fourier series, or simply the Fourier seris, of $\phi(x)$ on the interval $(-l, l)$ is defined as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}).$$

Exercise 4. Let m, n are positive integers. Prove the following formulas

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \quad forall \quad n, m$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0 \quad for \quad n \neq m$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \quad for \quad n \neq m$$

$$\int_{-l}^{l} 1 \cdot \cos\frac{n\pi x}{l}dx = 0 = \int_{-l}^{l} 1 \cdot \sin\frac{n\pi x}{l}dx$$

$$\int_{-l}^{l} \cos^2\frac{n\pi x}{l}dx = l = \int_{-l}^{l} \sin^2\frac{n\pi x}{l}dx$$

$$\int_{-l}^{l} 1^2 dx = 2l.$$

So the coefficients of the full Fourier series are

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}dx \qquad (n = 0, 1, 2, \cdots)$$

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}dx. \qquad (n = 1, 2, 3, \cdots)$$

Example 5. Let $\phi(x) \equiv x$ in the interval $(0, l)$. Its Fourier sine series has the coefficients

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x\sin\frac{m\pi x}{l}dx \\ &= (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

Thus in $(0, l)$ its Fourier sine series is

$$x = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots).$$

Its Fourier cosine series in $(0, l)$ has the coefficients

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l x dx = l, \\ A_m &= \frac{2}{l}\int_0^l x\cos\frac{m\pi x}{l}dx \\ &= \frac{2l}{m^2\pi^2}[(-1)^m - 1]. \end{aligned}$$

Thus in $(0, l)$ its Fourier cosine series is

$$x = \frac{l}{2} - \frac{4l}{\pi^2}(\cos\frac{\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi x}{l} + \cdots).$$

Its Fourier series in $(-l, l)$ has the coefficients

$$\begin{aligned} A_0 &= \frac{1}{l}\int_{-l}^l x dx = 0, \\ A_m &= \frac{1}{l}\int_{-l}^l x\cos\frac{m\pi x}{l}dx \\ &= \frac{x}{m\pi}\sin\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\cos\frac{m\pi x}{l}|_{-l}^{l} \\ &= 0, \\ B_m &= \frac{1}{l}\int_{-l}^l x\sin\frac{m\pi x}{l}dx \\ &= (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

So in $(-l,l)$ its full Fourier series is

$$x = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots).$$

Example 6. Solve the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0 \qquad & 0 < x < l \\ u(0,t) = u(l,t) = 0 & \\ u(x,0) = x, u_t(x,0) = 0. & \end{aligned}$$

From previous Lectures, we know that $u(x,t)$ has an expansion

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}.$$

Differentiating with respect to time yields

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}(-A_n\sin\frac{n\pi ct}{l} + B_n\cos\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}.$$

Setting $t=0$, we have

$$u(x,0) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l},$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\sin\frac{n\pi x}{l}.$$

Because the Fourier sine series of x and 0 in $(0,l)$ are

$$x = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots)$$

and

$$0 = 0.$$

So $B_n = 0$ and $A_n = (-1)^{n+1}\frac{2l}{n\pi}$.
So the solution is

$$u(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$