THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Midterm Solution

1. The net present value of investment plan 2 is

$$\begin{aligned} NPV_1 &= -12000 + 3000(1.05)^{-1} + 6000(1.05)^{-2} + 7000(1.05)^{-3} + 8000(1.05)^{-4} \\ &= 8927.80. \end{aligned}$$

The net present value of investment plan 2 is

$$\begin{aligned} NPV_2 &= -12000 + 7000(1.05)^{-1} + 8000(1.05)^{-2} + 3000(1.05)^{-3} + 6000(1.05)^{-4} \\ &= 9450.63 \\ &> NPV_1. \end{aligned}$$

Investment plan 2 is better.

2. The net present value of the 15 year mortgage is

$$\begin{aligned} NPV &= \sum_{i=1}^{60} 15000(1 + 0.03/4)^{-i} \\ &= 722600.60 \end{aligned}$$

Let A be her new monthly payment, then

$$\begin{aligned} NPV &= \sum_{i=1}^{360} A(1 + 0.04/12)^{-i} \\ &= 209.46A \end{aligned}$$

Thus, we have $A = 722600.60/209.46 = 3449.81$. The new monthly payment is 3449.81.

3. $(e^r = 1.0)$The payoff of my friend's offer is

$$\begin{aligned} B(0) &= 0 \\ B(T) &= \begin{cases} -Q & \text{if } S_T > 120 \\ 110 + Q - S_T & \text{if } S_T \le 120 \end{cases} \end{aligned}$$

Suppose we construct a portfolio with arbitrage opportunity as follows: long a, b and c call with strike $100, 120$ and 140 respectively and long d cash. Without loss of generality we may assume $\Pi(0) = 0$, i.e. $d = -60a - 45b - 35c$

Then, the values of portfolio is

$$\begin{aligned}
&\Pi(0) = 0 \\
&\Pi(T) = a(S_T - 100)^+ + b(S_T - 120)^+ + c(S_T - 140)^+ + 1.00d + B(T) \\
&\qquad = \begin{cases} -60a - 45b - 35c + Q - S_T + 110 & \text{if } S_T < 100 \\ -160a - 45b - 35c + Q + (a-1)S_T + 110 & \text{if } 100 \le S_T < 120 \\ -160a - 165b - 35c - Q + (a+b)S_T & \text{if } 120 \le S_T < 140 \\ -160 - 165b - 175c - Q + (a+b+c)S_T & \text{if } S_T \ge 140 \end{cases} \\
&\ \ \ge 0
\end{aligned}$$

By considering the last inequality and $S_T \to \infty$, we must have $a+b+c \ge 0$. Since the above system of inequality is piece-wise linear in S_T, it suffices to consider $\Pi(T)$ at the critical point, i.e. $S_T = 100, 120, 140$ and $S_T \to 120^-$. Thus, we have

$$\begin{cases} -60a - 45b - 35c + Q + 10 & \ge 0 \\ -40a - 45b - 35c + Q - 10 & \ge 0 \\ -40a - 45b - 35c - Q & \ge 0 \\ -20a - 25b - 35c - Q & \ge 0 \\ a + b + c & \ge 0 \end{cases}$$

If (a, b, c) satisfies $a + b + c > 0$ and gives arbitrage profit, one may always choose a smaller c. Thus, without loss of generality, we may assume $c = -a - b$. Then, we have

$$\begin{cases} -25a - 10b + Q + 10 & \ge 0 \\ -5a - 10b + Q - 10 & \ge 0 \\ -5a - 10b - Q & \ge 0 \\ 15a + 10b - Q & \ge 0 \end{cases}$$

Then $a = 1, b = -1, Q = 5$ satisfy above inequality and when $S_T < 100$, $\Pi(T) > 0$.

4. We construct a portfolio: long 1 chooser option at time t. Denotes $C_h(t)$ be the price for that option at time t. Then, the values of the portfolio is

$$\begin{aligned}\Pi_1(t) &= C_h(t)\\ \Pi_1(T_1) &= \max\{C_1(T_1,T_2), P_1(T_1,T_2)\}\\ &= \begin{cases} C_1(T_1,T_2) & \text{if } S(T_1) \geq Ke^{-r(T_2-T_1)}\\ P_1(T_1,T_2) & \text{if } S(T_1) < Ke^{-r(T_2-T_1)} \end{cases},\end{aligned}$$

by considering the call-put parity

$$C_1(T_1,T_2) - P_1(T_1,T_2) = S(T_1) - Ke^{-r(T_2-T_1)},$$

where $C_1(T_1,T_2)$ and $P_1(T_1,T_2)$ denotes the price of European call and put option with same strike K and same maturity T_2.

Now, we construct another portfolio: long 1 European call option with strike K and maturity T_2 and long 1 European put option strike $Ke^{-r(T_2-T_1)}$ and maturity time T_1 at time t. Denotes their prices by $C_1(t,T_2)$ and $P_2(t,T_1)$ respectively. Then, the values of the portfolio are

$$\begin{aligned}\Pi_2(t) &= C_1(t,T_2) + P_2(t,T_1)\\ \Pi_2(T_1) &= C_1(t,T_2) + (Ke^{-r(T_2-T_1)} - S(T_1))^+\\ &= \begin{cases} C_1(T_1,T_2) & \text{if } S(T_1) \geq Ke^{-r(T_2-T_1)}\\ C_1(T_1,T_2) + Ke^{-r(T_2-T_1)} - S(T_1) & \text{if } S(T_1) < Ke^{-r(T_2-T_1)} \end{cases}\\ &= \begin{cases} C_1(T_1,T_2) & \text{if } S(T_1) \geq Ke^{-r(T_2-T_1)}\\ P_1(T_1,T_2) & \text{if } S(T_1) < Ke^{-r(T_2-T_1)}, \end{cases}\end{aligned}$$

where the last equality comes from the call-put parity. Since $\Pi_1(T_1) = \Pi_2(T_1)$, we have $\Pi_1(t) = \Pi_2(t)$ for all $t < T_1$ by the replication argument. The assertion now follows.

5.
$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{1.05 - 0.9}{1.15 - 0.9} = 0.6$$

The risk neutral probability measure $\mathbb{Q}$ is given by

$$\begin{cases} \mathbb{Q}[S_{t_1} = uS_0] = \mathbb{Q}[f(t_1) = f_u] & = 0.6\\ \mathbb{Q}[S_{t_1} = dS_0] = \mathbb{Q}[f(t_1) = f_d] & = 0.4. \end{cases}$$

Note $S_{uu} = 100(1.15)^2 = 132.25$, $S_{ud} = S_{du} = 100(1.15)(0.9) = 103.5$, $S_{dd} = 100(0.9)^2 = 81$. To compute the price of a European put option, we have

$$\begin{aligned}
f_{uu} &= (100 - 132.25)^+ = 0 \\
f_{ud} = f_{du} &= (100 - 103.5)^+ = 0 \\
f_{dd} &= (100 - 81)^+ = 19 \\
f_u &= (1.05)^{-1}(0.6 \times 0 + 0.4 \times 0) = 0 \\
f_d &= (1.05)^{-1}(0.6 \times 0 + 0.4 \times 19) = 7.24 \\
f &= (1.05)^{-1}(0.6 \times 0 + 0.4 \times 7.24) = 2.76.
\end{aligned}$$

The price of the European put option is 2.76.

The dynamic trading strategy is given by

$$\begin{aligned}
\phi_0 &= \frac{0 - 7.24}{100(1.15 - 0.9)} = -0.29 \\
\phi_1^d &= \frac{0 - 19}{100(1.15 \times 0.9 - 0.9^2)} = -0.84 \\
\phi_1^u &= \frac{0 - 0}{100(1.15^2 - 1.15 \times 0.9)} = 0.
\end{aligned}$$

The price of the European call option is given by

$$\begin{aligned}
C_E &= S_0 - Ke^{-r(2-0)} + P_E \\
&= 100 - 100(1.05)^{-2} + 2.76 \\
&= 12.05
\end{aligned}$$