MATH4210: Financial Mathematics Tutorial 1

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16 September, 2020

Interest Rate

Let r be the interest rate. Suppose that you place $\$x_0$ in an account in a bank. After n years, you will have the amount

- $y_n = x_0(1 + nr)$ if the interest rate is the simple interest rate.
- $y_n = x_0(1 + r)^n$ if the interest rate is the annual compound interest rate.
- $y_n = x_0(1 + \frac{r}{m})^{mn}$ if the interest rate is the compound interest rate and compound m times per annul.
- $y_n = x_0 e^{nr}$ if the interest rate is the continuous compound interest rate.

Interest Rate

Question

a) Find the value of a 10-year zero-coupon bond of face value \$100 *if the the annual simple interest rate is* 2%.
b) Find the face value of a 10-year zero-coupon bond if it is issued for \$100 *and the continuous compound interest rate is* 3%.

Answer

a)

$$
\begin{aligned}
Value &= 100 \times (1 + 10 \times 2\%)^{-1} \\
&= 83.33
\end{aligned}
$$

b)

$$
\begin{aligned}
Face\ Value &= 100 \times e^{10 \times 0.03} \\
&= 134.99
\end{aligned}
$$

Interest Rate

Question

A bank offers a special plan for the new users. If you deposit money in the bank, you will receive dividend monthly with interest rate 4.95% *for a year. If you borrow money from the bank, you need to return to the bank after a year with annually interest rate* 5%. *Is there any arbitrage opportunity?*

Interest Rate

Answer

If you borrow \$1000 *and deposit* \$1000 *at* $T = 0$, *then the portfolio at* $t = 0$ *is*

$$\Pi(0) = 1000 - 1000 = 0.$$

After a year,

$$\Pi(1) = 1000(1 + 0.0495/12)^{12} - 1000 \times 1.05 = 0.638 > 0.$$

This means that you could earn money from the bank without risk or work, i.e. arbitrage opportunity! In fact, no bank offers such a stupid plan.

Present Value

Since we can always use \$$x_0$ now as principal in a risk-free investment at (continuous compound interest) rate $r > 0$ guaranteeing the amount

$$x_0 e^{rt} > x_0$$

at time t. Equivalently, if we deposit \$$xe^{-rt}$ at the bank, we get \$$x$ at time t, thus

We call xe^{-rt} the present value (PV) of x,

which is also called the discounted value of x at the future time t, and the factor e^{-rt} is called the discount factor.

Present Value

Similarly, for discrete compound interest r, where one compounds k times during $(0, t]$ at annual rate r, the present value (PV) of x at time t is

$$x(1 + rt/k)^{-k},$$

where $(1 + rt/k)^{-k}$ is the discount factor.

Present Value

Question

Pricing a coupon bond: consider a 2-year \$2000 bond, that has coupons every 1/2 year in the amount of \$50, for a total of four times until 2 years at which time you receive \$2050. Suppose the continuous compound interest rate is r. What is its net present value (NPV)?

Answer

- *The first payment of* 50 *has PV of* $50e^{-0.5r}$
- *The second payment of* 50 *has PV of* $50e^{-r}$
- *The third payment of* 50 *has PV of* $50e^{-1.5r}$
- *The last payment of* 2050 *has PV of* $2050e^{-2r}$
- *The net present value is* $50e^{-0.5r} + 50e^{-r} + 50e^{-1.5r} + 2050e^{-2r}$.
- *If* $r = 3\%$, $NPV \approx 2076$

Present Value

Question

For the expropriation of your ancestral grave, the government promises to pay you (and your descendant) $10,000 *immediately and the same amount every year perpetually. If the the compound annual interest rate is* 2.5%, *what is its net present value?*

Answer

$$\begin{aligned} NPV &= 10000 + 10000 \times 1.025^{-1} + 10000 + \times 1.025^{-2} + \dots \\ &= 10000 \times \sum_{i=0}^{\infty} 1.025^{-i} \\ &= 10000 \times \frac{1.025}{1.025 - 1} \\ &= 410000 \end{aligned}$$

Internal Rate of return

Given a cash flow stream $(x_0, x_1, \ldots, x_n)$, the associated Internal Rate of Return (IRR) r_{irr} is the rate under which the corresponding NPV equals to 0, i.e. r_{irr} is the solution to the equation

$$\sum_{i=0}^{n} x_i (1 + r_{irr})^{-i} = 0,$$

if the interest is annually compounded. r_{irr} is the solution to the equation

$$\sum_{i=0}^{n} x_i e^{-i r_{irr}} = 0,$$

if the interest is continuously compounded. Generally, this equation cannot be solved by algebraic method for $n \geq 5$. We will employ Newton's method to solve it numerically in our next example.

Newton's Method

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a *nice* function. Recall that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

when x is close to x_0. To solve $f(x) = 0$ numerically, we obtain

$$x \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

This suggests a recurrence relation to solve $f(x) = 0$: start from an initial guess x_0, we can approximate the solution by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached.

Bond Yield

Question

Consider a 2-year \$2000 *bond, that has coupons every* $1/2$ *year in the amount of* \$50, *for a total of four times until* 2 *years at which time you receive* \$2050. *The bond price is* \$2100. *What is the yield (i.e. internal rate of return) if the rate is continuously compound?*

Bond Yield

Answer

The cash flow stream is $(-2100, 50, 50, 50, 2050)$*. The yield is the solution to the following:*

$$-2100 + 50e^{-0.5\lambda} + 50e^{-\lambda} + 50e^{-1.5\lambda} + 2050e^{-2\lambda} = 0.$$

Let $x = e^{-0.5\lambda}$*, the above equation is equivalent to a degree 4 polynomial:*

$$f(x) = 2050x^4 + 50x^3 + 50x^2 + 50x - 2100 = 0.$$

Note $f'(x) = 8200x^3 + 150x^2 + 100x + 50$*. Apply Newton's method with initial guess* $x_0 = 1$,
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.9882, x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.9880, \ldots$ *This yields* $x \approx 0.988026$,, *i.e.* $\lambda \approx 0.024$