THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Assignment 2
Due date: 16 October 2020 11:59 p.m.

Please submit this assignment on blackboard. If you have any questions regarding this assignment, please email your TA Wong Wing Hong (whwong@math.cuhk.edu.hk).

1. Suppose the continuous compounding interest rate is r and the price of a stock is $S(t)$ at time t. If it pays dividend $d \times S(t_D)$ at time t_D, where $0 < t_D < T$ and $0 < d < 1$, show that its forward price $F(0,T)$ satisfies $F(0,T) = \frac{1}{1+d} S(0)e^{rT}$ under no arbitrage opportunity assumption.

   Solution: We construct a portfolio $\Pi_1(t)$ as follows: long 1 stock at time 0 and use the dividend $dS(t_D)$ to long d stock at time t_D. Then,

$$\begin{aligned} \Pi_1(0) &= S(0) \\ \Pi_1(t_D) &= (1+d)S(t_d) \\ \Pi_1(T) &= (1+d)S(T) \end{aligned}$$

   Similarly, we construct another portfolio $\Pi_2(t)$: we long $(1+d)$ forward contract at time 0 and deposit $(1+d)F(0,T)e^{-rT}$ cash in the bank. Then

$$\begin{aligned} \Pi_2(0) &= (1+d)F(0,T)e^{-rT} \\ \Pi_2(T) &= (1+d)F(0,T) + (1+d)\big(S(T) - F(0,T)\big) = \Pi_1(T) \end{aligned}$$

   By the law of one price, we have $\Pi_1(0) = \Pi_2(0)$, i.e.

$$F(0,T) = \frac{1}{1+d} S(0)e^{rT}.$$

2. (Put-Call Parity Relation with Dividend) Assume that the value of the dividends of the stock paid during $[t,T]$ is a deterministic constant D at time $t_D \in (t,T]$. Let $S(t)$ be the stock price, r be the continuous

compounding interest rate, $C_E(t,K)$ and $P_E(t,K)$ be the prices of European call and put option at time t with strike K and maturity T respectively. Show that

$$C_E(t,K) - P_E(t,K) = S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)}$$

for all $t < T$.

Solution: We construct a portfolio $\Pi_1(t)$ as follows: long 1 call option and short 1 put option at time 0. Then,

$$\begin{aligned} \Pi_1(t) &= C_E(t,K) - P_E(t,K) \\ \Pi_1(T) &= (S(t) - K)^+ - (K - S(t))^+ = S(t) - K \end{aligned}$$

Similarly, we construct another portfolio $\Pi_2(t)$: long 1 stock at time 0 and short $Ke^{-r(T-t)} + De^{-r(t_D-t)}$ cash at time t, then use the dividend D to pay back the bank at time t_D. Then,

$$\begin{aligned} \Pi_2(t) &= S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)} \\ \Pi_2(t_D) &= S(t_D) + D - Ke^{-r(T-t_D)} - D = S(t_D) - Ke^{-r(t-t_D)} \\ \Pi_2(T) &= S(t) - K = \Pi_1(T) \end{aligned}$$

By the law of one price, we have $\Pi_1(t) = \Pi_2(t)$, i.e.

$$C_E(t,K) - P_E(t,K) = S(t) - Ke^{-r(T-t)} - De^{-r(t-t_D)}$$

3. Assume that the value of the dividends of the stock paid during $[t,T]$ is a deterministic constant D at time $t_D \in (t,T]$. Let $S(t)$ be the stock price, r be the continuous compounding interest rate, $C_A(t,K)$ and $P_A(t,K)$ be the prices of American call and put option at time t with strike K and maturity T respectively. Show that

$$C_A(t,K) - P_A(t,K) < S(t) - Ke^{-r(T-t)}$$

for all $t < T$.

Solution: Recall from tutorial 4 that $C_A(t,K) < C_E(t,K) + De^{-r(t_D-t)}$ and $P_A(t,K) \geq P_E(t,K)$. Thus, we have

$$\begin{aligned} C_A(t,K) - P_A(t,K) &< C_E(t,K) - P_E(t,K) + De^{-r(t_D-t)} \\ &= S(t) - Ke^{-r(T-t)} \end{aligned}$$

4. Suppose the continuous compounding interest rate is r, two European call options has same strike K and different maturity $T_1 < T_2$. Suppose the underlying asset pays a deterministic dividend D at $t_D \in (T_1, T_2]$. Prove

$$C_E(t,T_1) < C_E(t,T_2) + (De^{-r(t_D-t)} - (e^{-r(T_1-t)} - e^{-r(T_2-t)})K)^+$$

for all $t < T_1$.

Solution: By Q2, we have

$$\begin{aligned} C_E(T_1,T_2) &= S(T!) - Ke^{-r(T_2-T_1)} - De^{-r(t_D-T_1)} + P_E(T_1,T_2) \\ &> S(T_1) - Ke^{-r(T_2-T_1)} - De^{-r(t_D-T_1)}. \end{aligned}$$

Suppose

$$C_E(t_0,T_1) \geq C_E(t_0,T_2) + (De^{-r(t_D-t_0)} - (e^{-r(T_1-t_0)} - e^{-r(T_2-t_0)})K)^+$$

for some $t_0 < T_1$. We make a portfolio: long one call option with maturity T_2, short one call option with maturity T_1, and long $(De^{-r(t_D-t_0)} - (e^{-r(T_1-t_0)} - e^{-r(T_2-t_0)})K)^+$ cash at time t_0. Then, the values of the portfolio are

$$\begin{aligned} \Pi(t_0) &= C_E(t_0,T_2) - C_E(t_0,T_1) + (De^{-r(t_D-t_0)} - (e^{-r(T_1-t_0)} - e^{-r(T_2-t_0)})K)^+ \leq 0 \\ \Pi(T_1) &= C_E(T_1,T_2) - (S(T_1)-K)^+ + (De^{-r(t_D-T_1)} - (1-e^{-r(T_2-T_1)})K)^+ \\ &> \begin{cases} S(T_1) - Ke^{-r(T_2-T_1)} - De^{-r(t_D-T_1)} - (S(T_1)-K) + \\ \qquad De^{-r(t_D-T_1)} - (1-e^{-r(T_2-T_1)})K & \text{if } S(T_1) > K \\ 0 - 0 + 0 & \text{if } S(T_1) \leq K \end{cases} \\ &= 0 \end{aligned}$$

This is an arbitrage opportunity. Thus, we must have

$$C_E(t,T_1) < C_E(t,T_2) + (De^{-r(t_D-t)} - (e^{-r(T_1-t)} - e^{-r(T_2-t)})K)^+$$

5. Suppose that we have the following 3 European call options with the same maturity T in the financial market:

Type	Strike Price	Price at time 0
Call	90	15
Call	100	12
Call	110	5

Suppose that the continuous compounding interest rate is $r = 0$ in the market and the maturity time is $T = 1$. Can you construct an arbitrage portfolio with the above options?

Solution: No, suppose we long a, b, c options with strike 90, 100, 110 respectively, and long d cash (here long -1 means short 1). Suppose this portfolio gives us arbitrage opportunity. Then the values of the portfolio are

$$\Pi(0) = 15a + 12b + 5c + d$$

$$\begin{aligned}\Pi(1) &= a(S(1) - 90)^+ + b(S(1) - 100)^+ + c(S(1) - 110)^+ + d \\ &= \begin{cases} d & \text{if } S(1) \leq 90 \\ aS(1) - 90a + d & \text{if } 90 < S(1) \leq 100 \\ (a+b)S(1) - 90a - 100b + d & \text{if } 100 < S(1) \leq 110 \\ (a+b+c)S(1) - 90a - 100b - 110c + d & \text{if } S(1) \geq 110 \end{cases}\end{aligned}$$

To have arbitrage profit, we have $\Pi(0) \leq 0$ and $\Pi(1) \geq 0$ with $\mathbb{P}(\Pi(1) > 0) > 0$. $\Pi(0) \leq 0$ is equivalent to

$$d \leq -15a - 12b - 5c$$

If (a, b, c, d) gives us an arbitrage profit, then $(a, b, c, -(15a+12b+5c))$ also gives us an arbitrage profit. Without loss of generality, we may assume $d = -15a - 12b - 5c$. Then the portfolio becomes

$$\Pi(0) = 0$$

$$\begin{aligned}\Pi(1) &= a(S(1) - 90)^+ + b(S(1) - 100)^+ + c(S(1) - 110)^+ - 15a - 12b - 5c \\ &= \begin{cases} -15a - 12b - 5c & \text{if } S(1) \leq 90 \\ aS(1) - 105a - 12b - 5c & \text{if } 90 < S(1) \leq 100 \\ (a+b)S(1) - 105a - 112b - 5c & \text{if } 100 < S(1) \leq 110 \\ (a+b+c)S(1) - 105a - 112b - 115c & \text{if } S(1) \geq 110 \end{cases}\end{aligned}$$

If $a + b + c < 0$ and $S(1) \to \infty$, then $\Pi(1) \to -\infty$, which contradicts our assumption. Thus, we must have $a + b + c \geq 0$. Also, since $\Pi(1)$ is piecewise linear in $S(1)$ and $a + b + c \geq 0$, to solve $\Pi(1) \geq 0$, it suffices to consider the critical point, i.e. $S(1) = 90, 100$ and 110, i.e., it suffices to consider

$$\begin{cases} -15a - 12b - 5c & \geq 0 \\ -5a - 12b - 5c & \geq 0 \\ 5a - 2b - 5c & \geq 0 \\ a + b + c & \geq 0, \end{cases}$$

which is equivalent to

$$\begin{cases} a & \leq -\frac{4}{5}b - \frac{1}{3}c \\ a & \leq -\frac{12}{5}b - c \\ a & \geq \frac{2}{5}b + c \\ a & \geq -b - c \end{cases}$$

Thus, we have

$$\begin{cases} -\frac{4}{5}b - \frac{1}{3}c & \geq \frac{2}{5}b + c \\ -\frac{4}{5}b - \frac{1}{3}c & \geq -b - c \\ -\frac{12}{5}b - c & \geq \frac{2}{5}b + c \\ -\frac{12}{5}b - c & \geq -b - c, \end{cases}$$

which is equivalent to

$$\begin{cases} b & \leq -\frac{10}{9}c \\ b & \geq -\frac{10}{3}c \\ b & \leq -\frac{5}{7}c \\ b & \leq 0, \end{cases}$$

where we obtain $c \geq 0$. Let $c = 1, b = -2, a = 1, d = 4$, i.e. long 1, short 2 , long 1 call options and long 4 cash, the values of the portfolio are

$$\begin{aligned} \Pi(0) &= 0 \\ \Pi(1) &= (S(1) - 90)^+ - 2(S(1) - 100)^+ + (S(1) - 110)^+ + 4 \\ &= \begin{cases} 4 & \text{if } S(1) < 90 \\ S(1) - 86 & \text{if } 90 \leq S(1) < 100 \\ 114 - S(1) & \text{if } 100 \leq S(1) < 110 \\ 4 & \text{if } S(1) \geq 110 \end{cases} \\ &> 0, \end{aligned}$$

which is an arbitrage profit.