THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Assignment 1 v2
Due date: 30 September 2020 11:59 p.m.

Please submit this assignment on blackboard. If you have any questions regarding this assignment, please email your TA Wong Wing Hong (whwong@math.cuhk.edu.hk).

1. Find the price of a bond with face value \$2,000 and \$10 annual coupons that matures in 4 years, given that the continuous compounding rate is a) 8%, b) 5%, c) 0%. What is the limit of this price as r goes to infinity?

   Solution:
   a. $P = 10e^{-0.08} + 10e^{-0.08\times 2} + 10e^{-0.08\times 3} + 2010e^{-0.08\times 4} = 1485.18$
   b. $P = 10e^{-0.05} + 10e^{-0.05\times 2} + 10e^{-0.05\times 3} + 2010e^{-0.05\times 4} = 1672.82.$
   c. $P = 10 + 10 + 10 + 2010 = 2040$
   When $r \to \infty$, $P = 10e^{-r} + 10e^{-2r} + 10e^{-3r} + 2010e^{-4r} \to 0.$

2. Suppose the discrete annual compound interest rate is 5% where ones compound monthly. Compute the corresponding continuous compounded interest rate r up to 6 decimal places.

   Solution:

$$e^r = (1 + \frac{0.05}{12})^{12}$$
$$r = 12\log(1 + \frac{0.05}{12}) \approx 0.049896$$

3. Suppose the price and the face value of a bond is both \$1,000. The bond matures in 5 years with annual coupon \$100. Find its a) bond yield by applying Newton's method with initial guess $x_0 = e^{-r_0} = 1$; b) duration; and c) convexity, if the interest rate is continuously compounded. (You may refer to tutorial 1 for Newton's method.)

Solution:
a. Let λ be the bond yield, we have

$$P = \sum_{i=1}^{5} c_i e^{-\lambda t_u}.$$

Note that $t_i = i$ for $i = 1, 2, 3, 4, 5$, $c_i = 100$ for $i = 1, 2, 3, 4$, $c_5 = 1000 + 100 = 1100$ and $P = 1000$. Let $x := e^{-\lambda}$, then we have

$$f(x) := 1100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x - 1000 = 0$$
$$f'(x) = 5500x^4 + 400x^3 + 300x^2 + 200x + 100x$$

Applying Newton's method to solve $f(x) = 0$,

$$x_0 = 1,$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{12}{13}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.909470\ldots$$

Repeatedly, this yields the solution $x^* \approx 0.9091, \lambda = -\log(x^*) = 0.095310$
b.

$$D = -\frac{1}{P}\frac{\partial P}{\partial \lambda} = \frac{1}{P}\sum_{i=1}^{5} t_i c_i e^{-\lambda t_i} = 4.169865$$

c.

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2} = \frac{1}{P}\sum_{i=1}^{5} t_i^2 c_i e^{-\lambda t_i} = 19.26583$$

4. (a) Suppose $x \neq 1$. Show that

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x},$$

for all positive integer n.

(b) Suppose $|x| < 1$. Show that

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}.$$

(c) Joyce wants to use a land to build a church. The government requires she to pay the nominal rent \$1,000 every year perpetually. A bank offer a plan: Joyce pay the bank \$50,000 at once and the bank promises to pay \$1,000 to the government every year. Suppose the discrete annual compound interest rate is 2%. Should Joyce accept this offer? (Note that you must pay the rent for the first year before using the land.)

Solution:

a.

$$\sum_{i=0}^{n} x^i = \sum_{i=0}^{n} x^i \times \frac{1-x}{1-x} = \frac{\sum_{i=0}^{n} x^i - x\sum_{i=0}^{n} x^i}{1-x} = \frac{\sum_{i=0}^{n} x^i - \sum_{i=1}^{n+1} x^i}{1-x} = \frac{1-x^{n+1}}{1-x}$$

b. Note that if $|x| < 1$,

$$\lim_{n\to\infty} x^{n+1} = 0.$$

Thus,

$$\sum_{i=0}^{\infty} x^i = \lim_{n\to\infty} \sum_{i=0}^{n} x^i = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}.$$

c. Net present value of the bank's offer with respect to Joyce is

$$\text{NPV} = -50000 + \sum_{i=0}^{\infty} 1000 \times 1.02^{-i} = -50000 + \frac{1000}{1-1.02^{-1}} = 1000 > 0.$$

Hence, Joyce should accept this offer.