MATH 4210 - Financial Mathematics

Black-Scholes Formula

Assumptions of Black-Scholes model

(1) Stock price $(S_t)_{0 \le t \le T}$ follows the Black-Scholes model:

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right),$$

or equivalently,

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

(2) Risk-free interest rate r is constant.

- Short selling is allowed.
- No transaction fees.
- All securities are perfectly divisible.
- No dividends during the lifetime of the derivatives.
- Security trading is continuous.
- No arbitrage opportunities.

Stochastic Integral: motivation

Dynamic trading: let $t_k := k\Delta t$, risky asset price $(S_{t_k})_{k\geq 0}$, interest rate $r \geq 0$.

Discrete time dynamic trading between t_k and t_{k+1}:

$$\begin{aligned}
\Pi_{t_{k+1}} &= \phi_{t_k} S_{t_{k+1}} + \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right) e^{r\Delta t} \\
&= \Pi_{t_k} e^{r\Delta t} + \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k} e^{r\Delta t}\right) \\
&\approx \Pi_{t_k} + \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right) r\Delta t + \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k}\right).
\end{aligned}$$

Then

$$\Pi_{t_n} = \Pi_0 + \sum_{k=0}^{n-1} \left(\Pi_{t_k} - \phi_{t_k} S_{t_k}\right) r\Delta t + \sum_{k=0}^{n-1} \phi_{t_k}\left(S_{t_{k+1}} - S_{t_k}\right).$$

The continuous time limit:

$$\Pi_T = \Pi_0 + \int_0^T (\Pi_t - \phi_t S_t) r dt + \int_0^T \phi_t dS_t.$$

Dynamic trading strategy

We say a portfolio is *self-financing* if

$$\mathrm{d}\Pi_t \;=\; (\Pi_t - \phi_t S_t) r \,\mathrm{d}t \;+\; \phi_t \,\mathrm{d}S_t,$$

where Π_t denotes the total wealth of the portfolio, ϕ_t denotes the number of the stocks in the portfolio, $\Pi_t - \phi_t S_t$ denotes the wealth invested in the risk-free asset.

Option pricing by replication

Let us consider the European call option with payoff $(S_T - K)_+$, if there is a self-financing portfolio Π such that

$$\Pi_T = (S_T - K)_+,$$

then the option price at time t is given by

$$\Pi_t.$$

Option price at initial time 0 is Π_0.

Black-Scholes Formula

Remember that

$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma B_t\big), \text{ (or equivalently } \mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t).$$

Let $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function, then by Itô's lemma, $Y_t = u(t, S_t)$ satisfies

$$\begin{aligned}
\mathrm{d}Y_t &= \mathrm{d}u(t, S_t) = \mathrm{d}u\big(t, S_0 \exp\big((\mu - \sigma^2/2)t + \sigma B_t\big)\big) \\
&= \Big(\partial_t u(t, S_t) + \mu S_t \partial_s u(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 u(t, S_t)\Big)\,\mathrm{d}t \\
&\quad + \partial_s u(t, S_t)\sigma S_t \,\mathrm{d}B_t \\
&= \Big(\partial_t u(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 u(t, S_t)\Big)\,\mathrm{d}t + \partial_s u(t, S_t)\,\mathrm{d}S_t
\end{aligned}$$

Black-Scholes Formula: Delta hedging

Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\partial_t u(t,s) \;+\; rs\partial_s u(t,s) \;+\; \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 u(t,s) \;-\; ru(t,s) \;=\; 0,$$

and $u(T,s) = (s-K)_+$.

Then with dynamic trading strategy $\phi_t = \partial_S u(t,S_t)$, and initial wealth $\Pi_0 = u(0,S_0)$, one has

$$\Pi_t \;=\; u(t,S_t),$$

since

$$\begin{aligned} d\Pi_t &= \phi_t \, \mathrm{d}S_t \;+\; (\Pi_t - \phi_t S_t) r \, \mathrm{d}t \\ &= \partial_s u(t,S_t) \, \mathrm{d}S_t \;+\; \Big(\partial_t u(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 u(t,S_t)\Big) \, \mathrm{d}t \\ &= \mathrm{d}u(t,S_t). \end{aligned}$$

Black-Scholes Formula: Delta hedging

Since $u(T,s) = (s-K)_+$, one has

$$\Pi_T \; = \; u(T, S_T) \; = \; (S_T - K)_+,$$

i.e. the perfect replication of the payoff of the call option.

(1) Solve the Black-Scholes PDE

$$\partial_t u(t,s) \; + \; rs\partial_s u(t,s) \; + \; \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} u(t,s) \; - \; ru(t,s) \; = \; 0,$$

with terminal condition $u(T,s) = (s-K)_+$.

(2) Construct a perfect replication portfolio Π, i.e. with initial wealth $\Pi_0 = u(0, S_0)$ and dynamic trading strategy $\phi_t = \partial_s u(t, S_t)$, one has

$$\Pi_T = (S_T - K)_+.$$

(3) The call option price is given by

$$\Pi_0 = u(0, S_0).$$

Black-Scholes Formula

Theorem 1.1

The price of a vanilla European call option C_E is the solution of PDE

$$\begin{cases} \frac{\partial C_E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_E}{\partial S^2} + rS\frac{\partial C_E}{\partial S} - rc = 0, \\ C_E(T,S) = (S-K)^+. \end{cases}$$

Black-Scholes Formula

Theorem 1.2

The price of a vanilla European option with payoff $g(S_T)$ is the solution of PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \\ u(T, s) = g(s). \end{cases}$$

Continuous-Time Risk-Neutral Valuation

In the risk neutral world (under the risk neutral probability $\mathbb{Q}$), the stock price follows:

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t^{\mathbb{Q}}, \\ S_T &= S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B_T}, \\ \ln(S_T) &\sim^{\mathbb{Q}} N\left(\ln(S_0)+(r-\tfrac{1}{2}\sigma^2)T,\ \sigma^2 T\right). \end{aligned}$$

Proposition 1.1

One has

$$u(0, S_0) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} g(S_T)\right].$$

Continuous-Time Risk-Neutral Valuation

Continuous-Time Risk-Neutral Valuation

Black-Scholes Formula

Theorem 1.3

The the Black-Scholes formula for European call option is

$$C_E(0, S_0) = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

and the Black-Scholes formula for European put option is

$$P_E(0, S_0) = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, \mathrm{d}y$,

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Black-Scholes Formula

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More generally, one has:

Theorem 1.4

The the Black-Scholes formula for European call option is

$$C_E(t, S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2),$$

and the Black-Scholes formula for European put option is

$$P_E(t, S_t) = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Option Pricing

Monotonicity in the factors:

increasing in	call option price	intuitive reason
$S(t)$	increases	potential payoff increases
K	decreases	potential payoff decreases
$T - t$	increases	more *"time value"*
r	increases	present value of fees K decreases
volatility σ	increases	risk increases

We can justify these conclusions with the Black-Scholes formula.

Black-Scholes Formula

The Black-Scholes formula for the vanilla European call option has the following properties

- $\frac{\partial C_E}{\partial S} > 0$. (Note that $\Delta = \frac{\partial C_E}{\partial S}$.)
- $\frac{\partial C_E}{\partial K} < 0$.
- $\frac{\partial C_E}{\partial (T-t)} > 0$.
- $\frac{\partial C_E}{\partial r} > 0$.
- $\frac{\partial C_E}{\partial \sigma} > 0$.

Greek Letters

We define the Greek letters for an option or portfolio C_E as

- Delta: $\Delta = \frac{\partial C_E}{\partial S}$.
- Theta: $\Theta = \frac{\partial C_E}{\partial t}$.
- Gamma: $\Gamma = \frac{\partial^2 C_E}{\partial S^2}$.
- Rho: $\rho = \frac{\partial C_E}{\partial r}$.
- Vega: $\mathcal{V} = \frac{\partial C_E}{\partial \sigma}$.

Greek Letters

Because the price C_E satisfies

$$\frac{\partial C_E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_E}{\partial S^2} + rS\frac{\partial C_E}{\partial S} - rC_E = 0,$$

we derive that

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = rC_E.$$

Carr-Madan formula

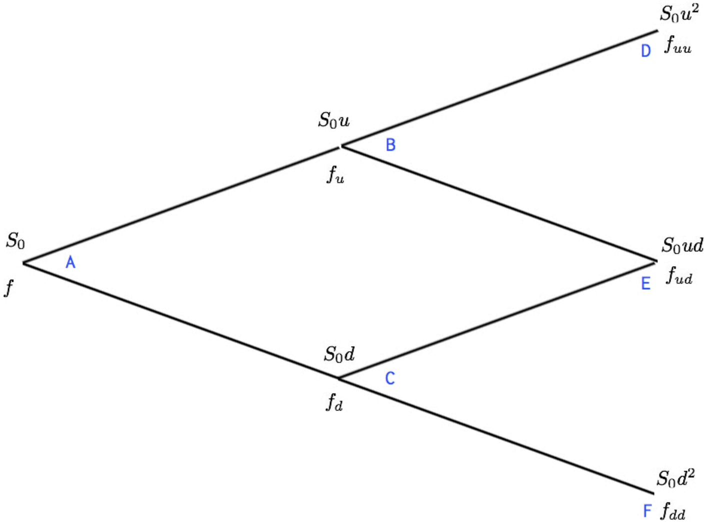
Let $g \in C^2$, it follows by Taylor's theorem that

$$\begin{aligned} g(S_T) &= g(F_0) + g'(F_0)(S_T - F_0) \\ &\quad + \int_0^{F_0} g''(K)(K - S_T)_+ dK + \int_{F_0}^{\infty} g''(K)(S_T - K)_+ dK. \end{aligned}$$

In practice, let us choose $F_0 = S_0 e^{rT}$, then the price (at time 0) of the option with payoff $g(S_T)$ is given by

$$g(F_0)e^{-rT} + \int_0^{F_0} g''(K) P_E(K) dK + \int_{F_0}^{\infty} g''(K) C_E(K) dK.$$

Pricing by the martingale approach: discrete time market



Pricing by the martingale approach: discrete time market

The risk-neutral probability

$$q = \frac{e^{r\Delta t} - d}{u - d}.$$

Price of the derivative option:

$$\begin{aligned}
f_u &= e^{-r\Delta t}(qf_{uu} + (1-q)f_{ud}) = \mathbb{E}^{\mathbb{Q}}[e^{-r\Delta t} f_{t_2} | S_{t_1} = S_u], \\
f_d &= e^{-r\Delta t}(qf_{ud} + (1-q)f_{dd}) = \mathbb{E}^{\mathbb{Q}}[e^{-r\Delta t} f_{t_2} | S_{t_1} = S_d], \\
f_{t_0} &= e^{-r\Delta t}(qf_u + (1-q)f_d) \\
&= e^{-2r\Delta t}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}) \\
&= \mathbb{E}^{\mathbb{Q}}[e^{-2r\Delta t} f_{t_2} | S_{t_0} = S_0],
\end{aligned}$$

It follows that the following discounted process are martingales:

$$\left(e^{-kr\Delta t} S_{t_k}\right)_{k=0,1,2}, \qquad \left(e^{-kr\Delta t} f_{t_k}\right)_{k=0,1,2}.$$

Pricing by the martingale approach: continuous time

Under the risk neutral probability $\mathbb{Q}$, the risky asset:

$$S_t = S_0 \exp\big((r - \sigma^2/2)t + \sigma B_t\big),$$

and the option price are given by

$$u(t, S_t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)} g(S_T) \big| S_t\big].$$

It follows that the following discounted process are martingales:

$$\big(e^{-rt} S_t\big)_{t\in[0,T]}, \qquad \big(e^{-rt} u(t, S_t)\big)_{t\in[0,T]}.$$

Self-financial strategy

Dynamic trading: let $t_k := k\Delta t$, risky asset price $(S_{t_k})_{k\geq 0}$, interest rate $r \geq 0$. Let us denote $\widetilde{S}_t := e^{-rt}S_t$ and $\widetilde{\Pi}_t := e^{-rt}\Pi_t$.

Discrete time dynamic trading between t_k and t_{k+1}:

$$\begin{aligned}\widetilde{\Pi}_{t_{k+1}} := e^{-rt_{k+1}}\Pi_{t_{k+1}} &= e^{-rt_{k+1}}\Big(\phi_{t_k}S_{t_{k+1}} + (\Pi_{t_k} - \phi_{t_k}S_{t_k})e^{r\Delta t}\Big) \\ &\approx \widetilde{\Pi}_{t_k} + \phi_{t_k}\big(\widetilde{S}_{t_{k+1}} - \widetilde{S}_{t_k}\big).\end{aligned}$$

Then

$$\widetilde{\Pi}_{t_n} = \widetilde{\Pi}_0 + \sum_{k=0}^{n-1} \phi_{t_k}\big(\widetilde{S}_{t_{k+1}} - \widetilde{S}_{t_k}\big).$$

The continuous time limit:

$$\begin{aligned}&\widetilde{\Pi}_T = \widetilde{\Pi}_0 + \int_0^T \phi_t d\widetilde{S}_t \\ \Longleftrightarrow\quad &\Pi_T = e^{rT}\Big(\Pi_0 + \int_0^T \phi_t d\widetilde{S}_t\Big).\end{aligned}$$

Self-financial strategy: martingale property

Then the portfolio is *self-financing* if

$$\begin{aligned} \Pi_t &= \Pi_0 + \int_0^t (\Pi_\theta - \phi_\theta S_\theta) r \, \mathrm{d}\theta \; + \; \int_0^t \phi_\theta \, \mathrm{d}S_\theta \\ \iff \quad \widetilde{\Pi}_t &= \Pi_0 + \int_0^t \phi_\theta \, \mathrm{d}\widetilde{S}_\theta . \end{aligned}$$

Recall that

$$d\widetilde{S}_\theta = \sigma \widetilde{S}_\theta dB_\theta^{\mathbb{Q}},$$

then

$$\widetilde{S}_t \; = \; S_0 + \int_0^t \sigma \widetilde{S}_\theta \, \mathrm{d}B_\theta^{\mathbb{Q}}, \qquad \widetilde{\Pi}_t \; = \; \Pi_0 + \int_0^t \phi_\theta \sigma \widetilde{S}_\theta \, \mathrm{d}B_\theta^{\mathbb{Q}},$$

are both martingales under $\mathbb{Q}$.

Pricing by the martingale approach

Theorem 1.5 (FTAP(Fundamental Theorem of Asset Pricing))

Assume that there exists a probability measure $\mathbb{Q}$ *equivalent to* $\mathbb{P}$*, such that the discounted values of all the financial assets are martingales under* $\mathbb{Q}$*. Then there is no arbitrage opportunity on the market.*