MATH4210: Financial Mathematics

I. Introduction

Financial Assets

The basic types of financial assets are *debt*, *equity*, and *derivatives*.

- *Debt* instruments are issued by anyone who borrows money - firms, governments, and households. The assets traded in debt markets, therefore, include corporate bonds, government bonds, residential and commercial mortgages, and consumer loans. These debt instruments are also called *fixed-income* instruments because they promise to pay fixed amout of cash in the future.
- *Equity* is the claim of the owners of a firm. Equity securities issued by corporations are called common stocks or shares. They are bought and sold in the stock markets.
- *Derivatives* are financial instruments that derive their value from the prices of one or more other assets such as equity securities, fixed-income securities, foreign currencies, or commodities. Their main function is to serve as tools for managing exposures to the risks associated with the underlying assets.

Financial Assets

Derivatives:
In particular, in the past few decades, we have witnessed the revolutionary period in the trading of financial derivative securities or contingent claims in financial markets around the world. It has just been mentioned above that a derivative (or derivative security) is a financial instrument whose value depends on the value of other more basic underlying variables.

The two most popular derivative securities are futures/forward contracts and options. They are now traded actively on many exchanges. Forward contracts, swaps, and many different types of options are regularly traded outside exchanges by financial institutions and their corporate clients in what we are termed the over-the-counter markets. Other more specialized derivatives often form part of a bond or stock issue.

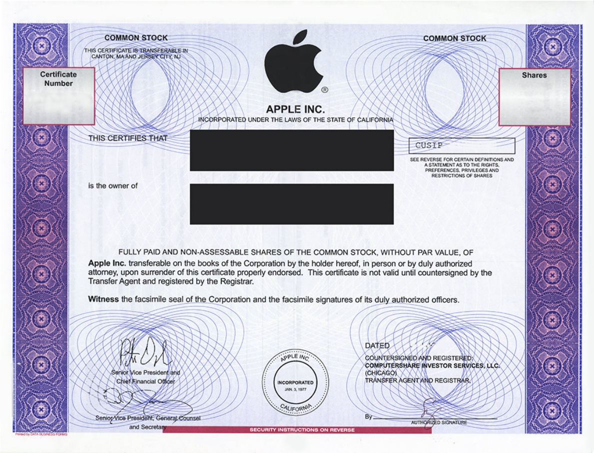
Financial Market

Stock markets:

If one has a bright idea for a new product or service, then he could raise capital to realize this idea by selling off future profits in the form of a stake in his new company. The investors of the company give him some cash, and in turn he gives them a contract stating how much of the company they own. Usually these investors are called shareholders. Shareholders may or may not receive dividends, depending on whether the company makes profit and decides to share this with its owners. What is the value of the company's stock? Its value reflects the views or predictions of investors about the likely dividend payments, future earnings, and resources that the company will control. These uncertainties are resolved (each trading day) by buyers and sellers of the stock. They exercise their views by trading shares in auction market. That is, most of the time a stock's value is judged by what someone else is willing to pay for it on a given day.

Financial Market

A sample of stock share



scource: http://patrickandmonica.net/archives/P2.html

Financial Market

Bond markets:

Bond is a kind of the fixed-income security. In its basic form a bond is a loan. It reflects a promise by a borrower, the seller of the bond, to repay the amount borrowed at a specific time, plus interest at an agreed-upon rate. The bond market is the channel through which governments and corporations that need to borrow money are matched with investors who have funds to lend. Such a market provides monetary liquidity that is vital to an economy's health. Every bond has a *face* or *par value*, which is the sum that the buyer of the bond will receive when the bond matures. When a bond is sold as a part of a new issue, its price is fixed. There are two major types of bonds - discount and coupon. A *discount*, or *zero-coupon bond* pays the owner only the face value of the bond at the time of maturity. A *coupon bond* pays the face value at maturity, and it also generates fixed, periodic payments known as coupons over the lifetime of the bond. For example, U.S. Treasury bonds have face values in multiples of 1000 dollars. Their maturity period can range from 2 years to 30 years. Typically the two-year bond is a coupon bond, whereas the 30-year Treasury bond is a zero-coupon bond.

Financial Market

A sample of government bond

scource: http://www.scripophily.net/chgogolobouc.html

Financial Market

Currency markets:

Currencies are bought and sold in currency market. What matters in such a dealing is the exchange rate. The fluctuation in exchange rates is unpredictable, however there is a link between exchange rates and the interest rates in the two countries.

For example, if the interest rate on dollars is higher than that on euro, we would expect to see euro depreciating against the dollars. Central banks can use interest rates as a tool for manipulating exchange rates, but only to a degree.

Financial Market

Commodity markets:
Physical assets such as oil, gold, copper, wheat or electricity are traded in the market. Recall that commodities are usually raw products such as metals, oil, food products, etc. The prices of these products are unpredictable but often show seasonal effects. Scarcity of the product results in higher price. Commodities are usually traded by people who have no need of the raw material. For example, they may just be speculating on the direction of gold without wanting to stockpile it or make jewelry. Most trading is done on the futures market, making deals to buy or sell the commodity at some time in the future.

Financial Market

Futures/forward and options markets:
Stock indices, forex (foreign exchange) and other non-physical assets are traded in the market. In fact, these are so-called derivatives. To be precise, a *stock derivative* is a specific contract whose value at some future date will depend entirely on the stock's future values. The person or firm who formulates this contract and offers it for sale is termed the writer. The person or firm who purchases the contract is termed the holder. The stock that the contract is based on is termed the underlying equity.

Financial Market

Ways derivatives are used

- To hedge risks
- To speculate (take a view on the future direction of the market)
- To lock in an arbitrage profit
- To change the nature of an investment without incurring the costs of selling one portfolio and buying another

Remark 1

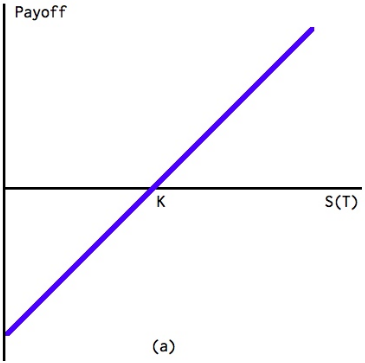
One is said to hedge a risk when reducing one's exposure to a loss entails giving up the possibility of a gain. For example, farmers who sell their future crops at a fixed price in order to eliminate the risk of a low price at harvest time give up the possibility of profiting from higher prices at harvest failures time. Forward contract could be used to hedge risk.

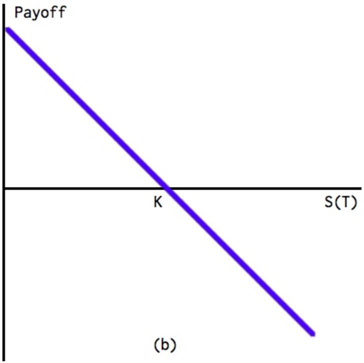
Forward Contract

A *forward contract* is an agreement where one party (long position) promises to buy a specified asset from another party (short position) at some specified time in the future and at some specified delivery price. No money changes hands until the delivery date or maturity of the contract. The terms of the contract make it an obligation to buy the asset at the delivery date, there is no choice in the matter. The asset could be a stock, a commodity, a currency or something else.

Forward Contract

Payoffs from forward contract at maturity: (a) long position, (b) short position.





Forward Contract

Example 1.1

Suppose a stock that pays no dividend is worth $\$60$. The annual compounding interest rate is 5%. What is the one-year forward price of the stock?

Forward Contract

Discussion

The answer is $60(1+5\%) = \$63$.

- If the forward price is more than $\$63$, say $\$67$, you could borrow $\$60$, buy one share of stock and sell one forward for $\$67$. Your net profit in one year will be $67 - 60(1+5\%) = \$4$.
- If the forward price is less than $\$63$, say $\$58$, you could short one share of stock to get $\$60$ and enter into one forward contract to buy the stock back for $\$58$ in one year. Your net profit in one year will be $60(1+5\%) - 58 = \$5$.

Forward Contract

Proposition 1.1 (without Dividend)

Suppose an investment asset that pays no dividend is worth S *now and the annual compounding interest rate is* r. *Then the* n*-year forward price of the stock is*

$$F = S(1+r)^n.$$

Forward Contract

Arbitrage opportunities when forward price is out of line with the spot price for asset without dividend

$F > S(1+r)^n$
action now:
1 Borrow S at interest r for n years 2 Buy one share asset 3 Short a n-year forward contract on the asset
action in n years:
1 Sell the asset for F 2 Repay the loan $S(1+r)^n$
net profit: $F - S(1+r)^n$

Forward Contract

Arbitrage opportunities when forward price is out of line with the spot price for asset without dividend

$F > S(1+r)^n$	$F < S(1+r)^n$
action now:	
1 Borrow S at interest r for n years 2 Buy one share asset 3 Short a n-year forward contract on the asset	1 Sell the asset for S 2 Invest S at interest r for n years 3 Long a n-year forward contract on the asset
action in n years:	
1 Sell the asset for F 2 Repay the loan $S(1+r)^n$	1 Buy the asset for F 2 Receive from the invest $S(1+r)^n$
net profit: $F - S(1+r)^n$	net profit: $S(1+r)^n - F$

Futures Contract

A *futures contract* is a standardized contract between two parties to buy or sell a specified asset of standardized quantity and quality at a specified future date at a price agreed today (the futures price). The contract is traded on a futures exchange. The profit or loss from the futures position is calculated every day and the change in this value is paid from one party to the other. Thus with futures contracts there is a gradual payment of funds from initiation until maturity.

Futures Contract

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Futures Contract

When the delivery period is reached, the futures price is equal to - or every close to - the spot price.

futures price is above spot price	futures price is below spot price
action at delivery date:	
1. Short a futures contract 2. Buy one share asset 3. Make delivery	1. Long a futures contract 2. Short one share asset 3. Make delivery
net profit: futures price−spot price	net profit: spot price−futures price

Comparison of Forward and Futures Contracts

Forward contract	*Futures contract*
Private contract between 2 parties	Traded on an exchange
Not standardized	Standardized contract
Usually one specified delivery date	Range of delivery dates
Settled at end of contract	Settled daily
Delivery or final cash settlement usually takes place	Contract is usually closed out prior to maturity
Some credit risk	Virtually no credit risk

(Vanilla) Options

A vanilla (call/put) *option* gives the holder (long position) of it the right to trade a prescribed asset (underlying) in the future at a previously agreed price.

- Expiry (maturity) date: the prescribed time to trade
- Call option: which gives the holder the right to buy the asset
- Put option: which gives the holder the right to sell the asset
- Strike (exercise) price: the previously agreed price to trade
- European option: which can only be exercised at the maturity date
- American option: which can be exercised at any time before and on the maturity date

Remark 2

The other party of the contract, who is known as the writer (short position) , does have a potential obligation: she/he must sell the asset if the holder chooses to exercise a call option.

Vanilla options

A sample of option

scource: http://www.scripophily.net/chcoca19.html

Vanilla options

One of our main concerns is:

- How much would one pay for the right to buy or sell an asset, i.e., what is the value of an option?

Vanilla options

Example 1.2

Fix a stock, say XXX. Denote by t the time of this moment. Let us also introduce some other notations. Let T be the maturity date, K be the strike price, $S(t)$ be the stock's market price at time t, and r be the annual simple interest rate. We would like to pay certain amount of money, say $c(t)$, so that at time T we are allowed (but not obligated) to buy one share XXX at the price K. What is $c(t)$?

Vanilla options

Discussion

Suppose we are holding a call option. There are two possibilities at time T.

- If $K < S(T)$, then the price K that we are allowed to buy the share is cheaper than the market price $S(T)$ at T . So, in this case, we can make profit by purchasing one share at K and selling it immediately at price $S(T)$. Hence we can make a profit $S(T) - K > 0$.
- We just do nothing if we see $S(T) \leqslant K$ at time T.

Vanilla options

Discussion (cont'd)

Notice that in both two cases, we will not lose money. Moreover, the option can bring some profit when $K < S(T)$, i.e., the value of the option at maturity is

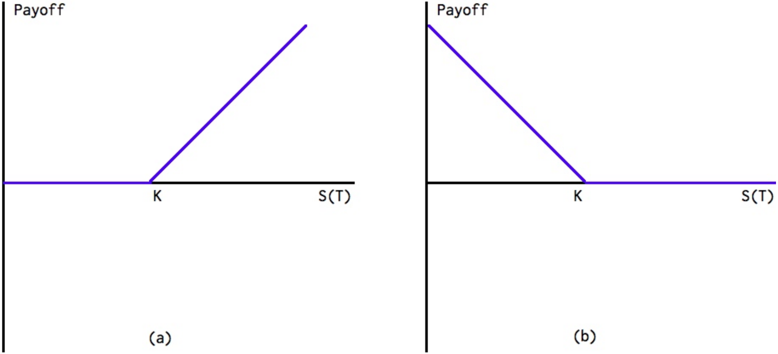
$$c(T) = \max\{S(T) - K, 0\}$$

This function of the underlying asset is called *payoff function*. The 'max' function represents the optionality. Sometimes, in notation we use $(S(T) - K)^+ := \max\{S(T) - K, 0\}$ and $c(T) = \max\{S(T) - K, 0\}$ is called boundary condition.

This product has a value. The question is: how much is the price $c(t)$? Is it $c(t) = \max\{S(t) - K, 0\}$?

Vanilla options

Payoffs from options at maturity date: (a) long a call option, (b) long a put option.



Vanilla options

In-the-money:
A term referring to a call (put) option when the market value of the underlying asset on which the option is written is greater (less) than the strike price.

Out-of-the-money:
A term referring to a call (put) option when the market value of the underlying asset on which the option is written is less (greater) than the strike price.

At-the-money:
A term referring to an option when the market value of the underlying asset on which the option is written is equal to the strike price.

Option Pricing

Factors to be taken into account:

In order to determine the value of a call option, $c(t)$, one has to take into account the following factors

- $S(t)$, today's share price. It seems reasonable to infer that the higher the share price is now then the higher the price is likely to be in the future. Thus the value of a call option today depends on today's share price.
- K, the strike price. The lower the exercise price is, the less the investor has to pay on exercise, and hence the higher the option value should be.
- $T - t$, the time to maturity. Usually an option is to expire in a significant time in the future, i.e., $T - t$ is a fairly long period. Imagine just before an option is to expire, there is little time for the asset price to change. In that case, the price at expiry can be predicted with a fair degree of certainty. We can conclude that the call option price must also be a function of thetime to maturity.

Option Pricing

Factors to be taken into account:

- r, the interest rate. Note that interest rates also play a role in affecting an option value. The option price is usually paid for up-front at the opening of the contract, whereas the payoff, if any, does not come until later. The option price should reflect the income that would otherwise have been earned by investing the premium in the bank.
- D, the dividend of the asset. The stock price will decrease right after dividend are issued.
- Finally, the option value depends on the property of the randomness of the asset price, the volatility, which will be explained later.

Thus the value of a call option is in fact a function of several variables:

$$c(t) = c(t, S(t), K, r, D, T, \cdots).$$

Option Pricing

Monotonicity in the factors:

increasing in	call option price	intuitive reason
$S(t)$	increases	potential payoff increases
K	decreases	potential payoff decreases
$T-t$	increases	more *"time value"*
r	increases	present value of fees K decreases
D	decreases	$S(t)$ decreases
volatility	increases	risk increases

We will justify these conclusions when we drive the explicit formula for option price.

Portfolio

Portfolio:

A portfolio is a combination of buying and selling (or longing and shorting) certain financial products. That is, a portfolio is an investment plan or trading strategy.

Value of a Portfolio:

The value of a portfolio is the amount of money paid to the holder of the portfolio so that the holder can transfer the ownership of the portfolio to the third party.

Portfolio

Example 1.3

If my portfolio consists of holding one unit of share with the price $S(t)$ at time t and I wish to get out of the portfolio. I can find someone in the market to pay me $S(t)$. After that he becomes the holder of the portfolio and I have got out of the market. In this case, the value of the portfolio is just $\Pi(t) = S(t)$.

In particular, the evolution of the portfolio value is given by

$$\Pi(t + \delta t) - \Pi(t) \;=\; S(t + \delta t) - S(t),$$

or equivalently, we write

$$d\Pi(t) = dS(t).$$

Portfolio

Example 1.4

Long x share stock $S(t)$ and short y number of bonds $B(t)$.
The value for this portfolio at time t is $\Pi(t) = xS(t) - yB(t)$.

In particular, the evolution of the portfolio value is given by

$$\Pi(t+\delta t) - \Pi(t) \;=\; x\big(S(t+\delta t) - S(t)\big) \;+\; (-y)\big(B(t+\delta t) - B(t)\big),$$

or equivalently, we write

$$d\Pi(t) = xdS(t) - ydB(t).$$

Remark 3

One can adjust the number/position (x, y) at any time t. The portfolio is said to be self-financing if you do not add or withdraw money from the portfolio. (We will develop the precise definition of a self-financing portfolio later).

Arbitrage opportunity

Arbitrage opportunity:

An arbitrage opportunity is a trading opportunity that either (1) takes a negative amount of cash to enter (i.e. cash flow to your pocket is positive) and promises a non-negative payoff to leave; or (2) takes a non-positive amount of cash to enter (cash flow to you is either zero or positive) and promises a non-negative payoff with the possibility of a positive payoff when leaving.

Arbitrage opportunity: Example

Consider three merchants who are willing to buy and sell bags containing apples and oranges (all of identical size and quality) as follows:

	Bag content	Price
Merchant I	3 apples, 2 oranges	5 dollars
Merchant II	2 apples, 3 oranges	6 dollars
Merchant III	5 apples, 5 oranges	10 dollars

One can buy a bag of 5 apples and 5 oranges from Merchant III for 10 dollars, and sell it to Merchant I and II for 11 dollars.

Remark: i) Given the prices of Merchant I and II, the no-arbitrage price for Merchant III should be 11 dollars.

ii) Pricing product III by replication.

Option Pricing

Principle:

– There is no arbitrage opportunity in the financial market.
  In the future, we always assume that the no-arbitrage principle is correct. The reason is because once such an arbitrage opportunity exists, many investors use this opportunity to make profit without risks. Then finally such opportunity will disappear.

  Consequence: Option pricing by replication. Assume that there is a (self-financing) portfolio Π whose initial value is $\Pi(0)$ and whose final value $\Pi(T) = c(T) = (S(T) - K)^+$, then the price of the call option is given by

$$c(0) = \Pi(0).$$

Option Pricing

Assumptions:

- Investors are well informed.
  This assumption implies that once there is an arbitrage opportunity, all the investors know it immediately. Then each of them tries to take the advantage of the arbitrage opportunity. Finally such advantage disappears. How much time will it take from the moment people discover an arbitrage opportunity to the time at which the opportunity disappears? The answer is 0 (seconds)! This is essentially our assumption.
- There are no transaction costs and taxes in each trading.
  This assumption is very strong. For the time being, we assume that it is correct so as to simplify a bit the mathematics to be used.
- There is no buy-sell spread.
  Namely, at any moment, purchasing price and selling price of a product are the same. This assumption is not true in practice. The purchasing price is equal to or higher than the selling price in practice.
- The asset is divisible
  The number x of shares of product one buys could be any real

Option Pricing

Remark 4

- *All the above assumptions are very ideal. We can think that as people improve the physical conditions, e.g., computer, communication means, etc., these assumptions will become more and more realistic. What we are mainly concerned with is what the financial markets look like if all these assumptions are satisfied.*
- *The mathematics in this course will emphasize two financial concepts that have had a startling impact over the last few decades on the way the financial industry views derivative trading.*
- *We will emphasize investments that replicate equities, and we will explore mathematical models of how equities behave in the absence of arbitrage opportunities. The combination of these two concepts furnishes a powerful tool for finding prices. Details and examples will be gradually introduced in the follwoing.*

Short Sell

Short sell:

Borrow shares and immediately sell them out to the market, and in the future buy back the (same amount) shares and return them. This is a trading strategy that yields a profit when the price of a security goes down and a loss when it goes up.

Short Sell

Mechanics of short selling:

To explain the mechanics of short selling, we suppose that an investor contacts a broker to short 500 IBM shares. The broker immediately borrows 500 IBM shares from another client and sells them in the open market in the usual way, depositing the sale proceeds to the investor's account. Providing there are shares that can be borrowed, the investor can continue to maintain the short position for as long as desired. At some stage, however, the investor will choose to instruct the broker to close out the position. The broker then uses funds in the investor's account to purchase 500 IBM shares and replaces them in the account of the client from which the shares were borrowed. If at any time while the contract is open, the broker runs out of shares to borrow, the investor is what is known as short-squeezed and must close out the position immediately even though he or she may not be ready to do so.