THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Assignment 3 Solution

1. Apply Itô formula to represent following $(X_t)_{t\geq 0}$ as an Itô process, i.e., $X_t = X_0 + \int_0^t b_s\ ds + \int_0^t \sigma_s\ dB_s$ for some (b,σ).

   (a) $X_t = B_t$.

   (b) $X_t = B_t^2$.

   (c) $X_t = B_t^3$.

   (d) $X_t = \exp(-\frac{\sigma^2 t}{2} + \sigma B_t)$.

   (e) $X_t = \sin(B_t)$.

   Solution:
   (a) $X_t = 0 + \int_0^t\ dB_s$.

   (b)$X_t = 0 + \int_0^t 2B_s\ dB_s + \int_0^t\ dt$.

   (c)$X_t = 0 + \int_0^t 3B_s\ ds + \int_0^t 3B_s^2\ dB_s$.

   (d)$X_t = 1 + \int_0^t \sigma e^{-\frac{\sigma^2}{2}s+\sigma B_s}\ dB_s$.

   (e)$X_t = 0 + \int_0^t (\frac{-sin(B_s)}{2})\ ds + \int_0^t cos(B_s)\ dB_s$

2.

3. Let $f : [0,T] \to \mathbb{R}$ be a bounded continuous (deterministic) function and $f_n(t) = \sum_{i=0}^{n-1} \alpha_i 1_{(t_i,t_{i+1}]}(t)$ for some deterministic constants α_i and discrete time interval $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_n = T$. Let

$$I_n := \int_0^T f_n(t)\ dB_t \quad \text{and} \quad I := \int_0^T f(t)\ dB_t.$$

(a) Prove that $I_n = \sum\limits_{i=0}^{n-1} \alpha_i (B_{t_{i+1}} - B_{t_i})$ and

$$I_n \sim N\Big(0, \sum_{i=0}^{n-1} \alpha_i^2 (t_{i+1} - t_i)\Big) = N\Big(0, \int_0^T f_n(t)^2 \, dt\Big).$$

(b) Assume that $f_n \xrightarrow{L^2([0,T])} f$.
Prove that $I_n \xrightarrow{L^2(\Omega)} I$ and $I \sim N\Big(0, \int_0^T f(t)^2 \, dt\Big)$.

(c) Compute the law of the following random variables (defined by stochastic integration):

$$\int_0^T t \, dB_t, \qquad \int_0^T e^t \, dB_t, \qquad \int_0^T \cos(t) \, dB_t.$$

Solution:
(a)

$$\begin{aligned} I_n &= \int_0^T f_n(t) \, dB_t \\ &= \sum_i \int_{t_i}^{t_{i+1}} f_n(t) \, dB_t \\ &= \sum_i \int_{t_i}^{t_{i+1}} \alpha_i \, dB_t \\ &= \sum_i \alpha_i (B_{t_{i+1}} - B_{t_i}) \end{aligned}$$

Since $(B_{t_{k+1}} - B_{t_k})$ and $(B_{t_{i+1} - B_{t_i}})$ are independent for $i \neq j$ and $(B_{t_{k+1}} - B_{t_k}) \sim N(0, t_{k+1} - t_k)$, we have $I_n \sim N(0, \sum \alpha_i^2 (t_{i+1} - t_i))$ and

$$\begin{aligned} \int_0^T f_n(t)^2 \, dt &= \sum_i \int_{t_i}^{t_{i+1}} f_n(t)^2 \, dt \\ &= \sum_i \alpha_i^2 (t_{i+1} - t_i). \end{aligned}$$

(b)From $f_n \xrightarrow{L^2([0,T])} f$ (which means $\int_0^T |f_n(t) - f(t)|^2\ dt|^2 \to 0$), one has

$$E\Big[\int_0^T (f_n(t) - f(t))^2\ dt\Big] \to 0.$$

since f_n, f are deterministic functions. Then, by Itô isometry,

$$\begin{aligned} E[|I_n - I|^2] &= E[|\int_0^T (f_n(t) - f(t))\ dB_t|^2] \\ &= E[|\int_0^T (f_n(t) - f(t))^2\ dt] \\ &\to 0. \end{aligned}$$

So $I_n \to I$ in $L^2(\Omega)$. Since $I_n \sim N\Big(0, \int_0^T f_n(t)^2\ dt\Big)$, from Tutorial 7 P4, we have I is a normal random variable with $E[I] = 0$, and $Var(I) = lim(\int_0^T f_n(t)^2\ dt)$. By triangular inequality

$$\|f\|_{L^2(0,T)} - \|f - f_n\|_{L^2(0,T)} \leq \|f_n\|_{L2(0,T)} \leq \|f\|_{L^2(0,T)} + \|f - f_n\|_{L^2(0,T)}$$

and $f_n \overset{L^2([0,T])}{\to} f$, we have $\lim_{n\to\infty} \int_0^T f_n(t)^2\ dt \to \int_0^T f(t)^2\ dt$. Then $Var(I) = \int_0^T f(t)^2\ dt$.

(c) Since f bounded and continuous on $[0, T]$, f can can be approximated by some simple functions in $L^2([0, T])$. From (b) we have

$$\int_0^T t\ dB_t \sim N(0, \int_0^T t^2\ dt) = N(0, \frac{T^3}{3}).$$

$$\int_0^T e^t\ dB_t \sim N(0, \int_0^T e^{2t}\ dt) = N(0, \frac{e^{2T} - 1}{2}).$$

$$\int_0^T cos(t)\ dB_t \sim N(0, \int_0^T cos^2(t)\ dt) = N(0, \frac{T + sin(2T)/2}{2}).$$