THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4210 Financial Mathematics 2020-2021 T1
Assignment 3
Due date: 28 November 2020 11:59 p.m.

Please submit this assignment on blackboard. If you have any questions regarding this assignment, please email TA Yang Fan (fyang@math.cuhk.edu.hk).

1. Apply Itô formula to represent the following process $(X_t)_{t\geq 0}$ as an Itô process, i.e., $X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dB_s$ for some process $(b_s, \sigma_s)_{s\geq 0}$.
   (a) $X_t = B_t$.
   (b) $X_t = B_t^2$.
   (c) $X_t = B_t^3$.
   (d) $X_t = \exp(-\frac{\sigma^2 t}{2} + \sigma B_t)$.
   (e) $X_t = \sin(B_t)$.

2. Let $f : [0,T] \to \mathbb{R}$ be a bounded continuous (deterministic) function and $f_n(t) = \sum_{i=0}^{n-1} \alpha_i 1_{(t_i,t_{i+1}]}(t)$ for some deterministic constants α_i and discrete time interval $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_n = T$. Let

$$I_n := \int_0^T f_n(t)\, dB_t \quad \text{and} \quad I := \int_0^T f(t)\, dB_t.$$

   (a) Prove that $I_n = \sum_{i=0}^{n-1} \alpha_i (B_{t_{i+1}} - B_{t_i})$ and

$$I_n \sim N\Big(0, \sum_{i=0}^{n-1} \alpha_i^2 (t_{i+1} - t_i)\Big) = N\Big(0, \int_0^T f_n(t)^2\, dt\Big).$$

   (b) Assume that $f_n \xrightarrow{L^2([0,T])} f$.
   Prove that $I_n \xrightarrow{L^2(\Omega)} I$ and $I \sim N\Big(0, \int_0^T f(t)^2\, dt\Big)$.
   (c) Compute the law of the following random variables (defined by stochastic integration):

$$\int_0^T t\, dB_t, \qquad \int_0^T e^t\, dB_t, \qquad \int_0^T \cos(t)\, dB_t.$$