Math 3280 Tutorial 2.

Ex 1. Suppose a die is rolled continually until a 6 appears, at which point the experiment stops. Then

① what is the sample space?

② Let E_n denote the event that the experiment stops after rolling n times. what is E_n?

③ what is the event $\left(\bigcup_{n=1}^{\infty} E_n\right)^c$?

Solution: (1) For $k \geq 0$, $i_1, \cdots, i_k \in \{1,2,3,4,5\}$. Denote by $(i_1, i_2, \cdots, i_k, 6)$ the outcome that the j-th roll gives i_j, $j=1,\cdots,k$. and $(k+1)$-th roll gives 6.

$S = \{\text{stops at some point } (k+1)\} \cup \{\text{not stop within finite many rolls}\}$

$= \{(i_1, i_2, \cdots, i_k, 6) : i_1, i_2, \cdots, i_k \in \{1,2,\cdots,5\}, k \geq 0\} \cup \{(i_1, i_2, \cdots\cdots), i_k \in \{1,2,\cdots,5\}, k \geq 1\}$

(2) $E_n = \{\underbrace{(i_1, i_2, \cdots, i_{n-1}, 6)}_{A_n} : i_1, \cdots, i_{n-1} \in \{1,2,3,4,5\}\}$. $P(A_n) = \frac{5 \times 5 \cdots \times 5}{6^n} = \frac{5^{n-1}}{6^n}$

(3) the event that 6 does not appear within finite many rolls

Ex 2: A football consists 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determing roomates. The pairing is done at random.

(1) What is the probability that there are $\underbrace{\text{no offensive-defensive roommate pairs}}_{E_0}$?

(2) What is the probability that there are $2k$ offensive-defensive roommate pairs? $k=1,2,\cdots,10$. ?

Solution: (1) There are

$$|S| = \binom{40}{2,2,\cdots,2} \frac{1}{20!} = \frac{40!}{2^{20} \cdot 20!}$$

possible ways to divide the 40 player into 20. (not ordered).

$$|E_0| = \binom{20}{2,2,\cdots,2}_{\text{offesive players}} \frac{1}{10!} \cdot \binom{20}{\cdots}_{\text{defensive plays.}} \frac{1}{10!}$$

$$\underbrace{(2,2,\cdots,2)}_{10 \text{ groups}} \cdots \quad \underbrace{(2,2,\cdots,2)}_{10 \text{ groups}} \cdots$$

$$P(E_0) = \frac{|E_0|}{|S|}.$$

(2) E_k denote that there are $2k$ offensive – defensive roommate pairs. $k=1,2,\cdots,10$.

$$|E_k| = \underbrace{\binom{20}{2k}}_{\text{chose } 2k \text{ from offensive}} (2k)! \underbrace{\binom{20}{2k}}_{\text{choose } 2k \text{ from defensives}} \cdot \underbrace{\binom{20-2k}{2,2,\cdots,2}}_{10-k \text{ pairs}} \frac{1}{(10-k)!} \cdot \underbrace{\binom{20-2k}{2,\cdots,2}}_{10-k \text{ group}} \frac{1}{(10-k)!}$$

$$= \binom{20}{2k}\binom{20}{2k}\cdot(2k)!\left[\binom{20-2k}{2,\cdots,2}\frac{1}{(10-k)!}\right]^2$$

offesive : $1, 2, \cdots, 2k, \mid 2k+1, \cdots, 20$

defensive : $1, 2, \cdots, 2k, \mid 2k+1, \cdots, 20.$

$$(2k)(2k-1)\cdots 1 = (2k)!$$

$$P(E_k) = \frac{|E_k|}{|S|}.$$

Ex 3. A pair of dice is rolled until a sum of 5 or 7 appears.

Find the probability that a 5 occurs first.

Solution: E denote the event that a sum of 5 occurs first.

For $n \geq 1$, denote by E_n the event that the sum of n-th roll is 5 and the sum of k-th roll is neither 5 nor 7 for $k=1,\ldots,n-1$.

Then we have

$$E = \left(\bigcup_{n=1}^{\infty} E_n\right).$$

Observe that the right hand side is a disjoint union.

$$P(E) = \sum_{n=1}^{\infty} P(E_n)$$

Let (i,j) denote an outcome of a roll. Then

$$i+j=5 \iff (i,j) = \underbrace{(1,4),(2,3),(3,2),(4,1)}_{\to 4}.$$

$$i+j=7 \iff (i,j) = \underbrace{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)}_{6}$$

$$P(E_1) = \frac{4}{36} = \frac{1}{9}$$

$$P(E_2) = \frac{36-10}{36} \cdot \frac{4}{36} = \frac{13}{18} \cdot \frac{1}{9}$$

second roll is 5.

first roll is not 5, 7.

$$P(E_3) = \frac{26}{36} \cdot \frac{26}{36} \cdot \frac{4}{36} = \left(\frac{13}{18}\right)^2 \cdot \frac{1}{9}.$$

first roll

$\vdots$

$$P(E_n) = \frac{26}{36} \cdot \frac{26}{36} \cdots \frac{26}{36} \cdot \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9}.$$

first

(n-1)-th roll

n-th

$i+j=5$.

$$P(E) = \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^n \cdot \frac{1}{9} = \frac{1}{9} \cdot \frac{1}{1-\frac{13}{18}} \, 1 = \frac{2}{5}.$$