The Chinese University of Hong Kong
Department of Mathematics

MATH3280 Introductory Probability 2020-2021

Solution to Midterm Examination

1 (10pts)

(i) (5pts)Note that $P(E \cap F) = P(E)P(F)$ since E and F are independent,

$$\begin{aligned} P(E^c \cap F) &= P(F) - P(E \cap F) \text{ (by Inclusion-exclusion Formula)} \\ &= P(F) - P(E)P(F) \\ &= (1 - P(E))P(F) \\ &= P(E^c)P(F) \end{aligned}$$

Hence E^c and F are independent.

(ii) (5pts)Yes. Since E, F and G are independent,

$$\begin{aligned} P((E \cup F) \cap G) &= P(E \cap G) + P(F \cap G) - P(E \cap F \cap G) \text{ (by Inclusion-exclusion Formula)} \\ &= P(E)P(G) + P(F)P(G) - P(E)P(F)P(G) \text{ (by the independence)} \\ &= [P(E) + P(F) - P(E \cap F)]P(G) \\ &= P(E \cup F)P(G) \text{ (by Inclusion-exclusion Formula).} \end{aligned}$$

2 (10pts)

(a) (4pts)By independence,

$$P(A \cap B \cap C) = P(A)P(B)P(C) = \frac{1}{2} \times \frac{1}{6} \times \frac{1}{3} = \frac{1}{36}.$$

(b) (6pts)By the Inclusion-exclusion formula and independence,

$$\begin{aligned} &P(A \cup B \cup C) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C) \\ &= \frac{13}{18}. \end{aligned}$$

(Alternatively, by the De Morgan's law and independence,

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P(A^c \cap B^c \cap C^c) \\ &= 1 - (1 - P(A))(1 - P(B))(1 - P(C)) \\ &= \frac{13}{18}. \end{aligned}$$

Or

$$\begin{aligned}P(A\cup B\cup C) =&P(A)+P(B)+P(C)-P(A\cap B\cap C^c)-\\&P(A\cap B^c\cap C)-P(A^c\cap B\cap C)-2P(A\cap B\cap C)\\=&P(A)+P(B)+P(C)-P(A)P(B)(1-P(C))-\\&P(A)P(C)(1-P(B)))-P(B)P(C)(1-P(A)))-2P(A)P(B)P(C)\\=&\frac{13}{18}.)\end{aligned}$$

3 (10pts)

(a) (6pts)Since $P\{X=i\}=F(i)-\lim_{b\to i^-}F(b)$, we have

$$\begin{aligned}P\{X=1\}&=\frac{1}{2}-\lim_{b\to 1^-}\frac{b}{4}=\frac{1}{4}\\P\{X=2\}&=\frac{11}{12}-\lim_{b\to 2^-}(\frac{1}{2}+\frac{b-1}{4})=\frac{1}{6}\\P\{X=3\}&=1-\frac{11}{12}=\frac{1}{12}.\end{aligned}$$

(b) (4pts)By the definition of cumulative distribution function,

$$\begin{aligned}P\{1\le X<3\}&=P(X<3)-P(X<1)\\&=\lim_{b\to 3^-}F(b)-\lim_{b\to 1^-}F(b)\\&=\frac{11}{12}-\lim_{b\to 1^-}\frac{b}{4}\\&=\frac{2}{3}.\end{aligned}$$

4 (10pts)

The target event is that the first four balls are all red while the last one is black. Hence the following arguments work:

S.1 Let $X_1,\ldots,X_5$ be the color the five balls, then X_i takes value r(red) or b(black). By the total probability formula,

$$\begin{aligned}&P(X_1=\cdots=X_4=r,X_5=b)\\&=P(X_1=\cdots=X_4=r,X_5=b|X_1=r)P(X_1=r)\\&=P(X_1=\cdots=X_4=r,X_5=b|X_1=X_2=r)P(X_2=r|X_1=1)P(X_1=r)\\&\cdots\\&=\frac{50}{90}\times\frac{49}{89}\times\frac{48}{88}\times\frac{47}{87}\times\frac{40}{86}.\end{aligned}$$

S.2 By permutation, the number of total possible permuations is $\frac{90!}{(90-5)!}$. The number of the permuations for first four red balls is $\frac{50!}{(50-4)!}$ and that for the last black ball is $\frac{40!}{(40-1)!}$. Hence the chance is

$$\frac{\frac{50!}{(50-4)!} \times \frac{40!}{(40-1)!}}{\frac{90!}{(90-5)!}} = \frac{50}{90} \times \frac{49}{89} \times \frac{48}{88} \times \frac{47}{87} \times \frac{40}{86}.$$

S.3 By combination, the number of total possible combinations is $\binom{90}{5}$. The number of the combinations for first four red balls is $\binom{50}{4}$ and that for the last black ball is $\binom{40}{1}$. However, in the total $\binom{5}{1}$ possible combinations of four red balls and one black ball, only one of them satisfies the requirement. Hence the chance is

$$\frac{\binom{50}{4} \times \binom{40}{1}/\binom{5}{1}}{\binom{90}{5}} = \frac{50}{90} \times \frac{49}{89} \times \frac{48}{88} \times \frac{47}{87} \times \frac{40}{86}.$$

5 (10pts)

Note that the probability density function of X, $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

(a) (5pts)Firstly, the cumulative distribution function $F(t)$ of Y is

$$F(t) = P\{Y \le t\} = P\{2X + 1 \le t\} = P\{X \le \frac{t-1}{2}\}$$
$$= \int_{-\infty}^{\frac{t-1}{2}} f(x)dx = \int_{-\infty}^{\frac{t-1}{2}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

Then the probability density function $g(y)$ of Y is

$$g(y) = F'(y) = f(\frac{y-1}{2}) \cdot \frac{1}{2} = \frac{1}{2\sqrt{2\pi}}e^{-\frac{(y-1)^2}{8}}.$$

(Alternatively, according to the fact that the linear transformation of normal random variable is still normal random variable, one can firstly compute the $E(Y)$ and $\sigma(Y)$ and then write down the $g(y)$ directly.)

(b) (5pts)For $t > 0$, the cumulative distribution function $G(t)$ of Z is

$$G(t) = P\{Z \le t\} = P\{X^2 \le t\} = P\{-\sqrt{t} \le X \le \sqrt{t}\}$$
$$= \int_{-\sqrt{t}}^{\sqrt{t}} f(x)dx = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$$

Then the probability density function $h(z)$ of Z is

$$h(z) = G'(z) = f(\sqrt{z}) \cdot \frac{1}{2\sqrt{z}} - f(-\sqrt{z}) \cdot (-\frac{1}{2\sqrt{z}}) = \frac{1}{\sqrt{2\pi z}}e^{-\frac{z}{2}},$$

where $z > 0$. When $z \le 0$, $h(z) = 0$.

6 (20pts)

Let X be the color of the marble, then X takes values b(black) or w(white). Let Y be the box from which the marble is drawn, then Y takes values o(odd) or e(even).

(a) (10pts)By the total probability formula,

$$\begin{aligned} P(X=b) &= P(X=b|Y=o)P(Y=o) + P(X=b|Y=e)P(Y=e) \\ &= \frac{3}{8} \times \frac{1}{2} + \frac{2}{6} \times \frac{1}{2} = \frac{17}{48}. \end{aligned}$$

(b) (10pts)By the Bayesian formula and the total probability formula,

$$\begin{aligned} P(Y=o|X=w) &= \frac{P(X=w|Y=o)P(Y=o)}{P(X=w)} \\ &= \frac{P(X=w|Y=o)P(Y=o)}{P(X=w|Y=o)P(Y=o) + P(X=w|Y=e)P(Y=e)} \\ &= \frac{\frac{5}{8} \times \frac{1}{2}}{\frac{5}{8} \times \frac{1}{2} + \frac{4}{6} \times \frac{1}{2}} = \frac{15}{31}. \end{aligned}$$

7 (20pts)

(a) (8pts)Since the other two numbers should be picked from the remaining 9 numbers other than 1, we have

$$P = \frac{\binom{9}{2}}{\binom{10}{3}} = \frac{3}{10}.$$

(b) (12pts)Note that

$$P(X=k) = \begin{cases} \frac{\binom{10-k}{2}}{\binom{10}{3}} & 1 \le k \le 8 \\ 0 & k = 9, 10 \end{cases}$$

Then

$$E[X] = \sum_{k=1}^{10} kP(X=k) = \sum_{k=1}^{8} k \frac{\binom{10-k}{2}}{\binom{10}{3}} = \frac{11}{4} = 2.75\,.$$

8 (10pts)

S.1 Mathematical induction: when $n = 1$, $P(E_1) \ge P(E_1)$ holds. Assume the proposition holds for n, it suffices to check it for the case $n+1$: Note that $P(A \cup B) = P(A) + P(B) - P(A \cap B) \ge P(A) + P(B) - 1$,

$$\begin{aligned} P(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1}) &\ge P(E_1 \cap E_2 \cap \cdots \cap E_n) + P(E_{n+1}) - 1 \\ &\ge P(E_1) + \cdots + P(E_n) - (n-1) + P(E_{n+1}) - 1 \\ &= P(E_1) + \cdots + P(E_{n+1}) - n. \end{aligned}$$

S.2 Directly,

$$\begin{aligned} P(E_1 \cap E_2 \cap \cdots \cap E_n) &= 1 - P((E_1 \cap E_2 \cap \cdots \cap E_n)^c) \\ &= 1 - P(E_1^c \cup E_2^c \cup \cdots \cup E_n^c) \text{ (by De Morgan's law)} \\ &\geq 1 - \sum_{k=1}^{n} P(E_k^c) \text{ (by the sub-additivity of probability)} \\ &= 1 - \sum_{k=1}^{n} [1 - P(E_k)] \\ &= P(E_1) + \cdots + P(E_n) - (n-1). \end{aligned}$$