The Chinese University of Hong Kong
Department of Mathematics

MATH3280 Introductory Probability

Solutions to Assignment 7

Problems

p.379 Q75

Note that X is a Poisson random variable with parameter $\lambda = 2$ and Y is a binomial random variable with parameters $n = 10$ and $p = 3/4$.

(a)

$$\begin{aligned} P(X+Y=2) &= P(X=0,Y=2)+P(X=1,Y=1)+P(X=2,Y=0) \\ &= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + \left(2^2e^{-2}/2\right)(1/4)^{10} \\ &\approx 6.027\times 10^{-5}. \end{aligned}$$

(b)

$$\begin{aligned} P(XY=0) &= P(\{X=0\}\cup\{Y=0\}) \\ &= P(X=0)+P(Y=0)-P(X=0,Y=0) \\ &= e^{-2}+(1/4)^{10}-e^{-2}(1/4)^{10} \\ &\approx 0.1353 \quad . \end{aligned}$$

(c) Since X and Y are independent,

$$E(XY) = E(X)E(Y) = 2\left(10\cdot\frac{3}{4}\right) = 15.$$

Theoretic exercises

p.384 Q50

Compute the second derivative of Ψ.

$$\Psi''(t) = \frac{M''(t)}{M(t)} - \left(\frac{M'(t)}{M(t)}\right)^2.$$

Hence

$$\Psi''(t)|_{t=0} = \frac{M''(0)}{M(0)} - \left(\frac{M'(0)}{M(0)}\right)^2 = E\left(X^2\right) - E(X)^2 = \mathrm{Var}(X).$$

Problems

p.412 Q1

We have $\mu = \sigma^2 = 20$. By Chebyshev's inequality,

$$\begin{aligned} P(0 < X < 40) &= 1 - P(|X - \mu| \geq \sqrt{20}\sigma) \\ &\geq 1 - \frac{1}{20} = \frac{19}{20}. \end{aligned}$$

p.413 Q4

(a) By Markov's inequality,

$$P\left(\sum_{i=1}^{20} X_i > 15\right) \leq \frac{E\left(\sum_{i=1}^{20} X_i\right)}{15} = \frac{20}{15} = \frac{4}{3}.$$

(b) By the central limit theorem,

$$\begin{aligned} P\left(\sum_{i=1}^{20} X_i > 15\right) &= P\left(\sum_{i=1}^{20} X_i > 15.5\right) \\ &= P\left(\frac{\sum_{i=1}^{20} X_i - 20}{\sqrt{20}} > \frac{15.5 - 20}{\sqrt{20}}\right) \\ &\approx 1 - \Phi(-1.01) \\ &= \Phi(1.01) \\ &\approx 0.8438 \quad . \end{aligned}$$

Theoretic exercises

p.414 Q5

Let $X_1, X_2, \dots$ be independent Bernoulli random variables with mean x. Define

$$Z_n = \frac{X_1 + \cdots X_n}{n}.$$

By the weak law of large numbers, for each $\varepsilon > 0$,

$$P\{|Z_n - x| > \varepsilon\} \to 0 \text{ as } n \to \infty.$$

(Alternatively, using central limit theorem to compute the probability $P\{|Z_n - x| > \varepsilon\} = 2\Phi(-\frac{\varepsilon\sqrt{n}}{\sigma}) \to 0$ as $n \to \infty$.)

Since f defined on $[0, 1]$ is continuous, f is bounded. Applying *Excercise p.414 q8.4* with $c = x$ and $g = f$, we have

$$E[f(Z_n)] \to f(x) \text{ as } n \to \infty.$$

(Alternatively, set $h = |f - f(x)|$ on $[0,1]$, then h is continuous and bounded above by some contant M. For each $\varepsilon > 0$. By the continuity of h, $\exists \delta > 0$, $\forall |Z_n - x| \le \delta$, $h \le \frac{\varepsilon}{2}$. By the weak law of large numbers, $\exists N \in \mathbb{N}$ such that $\forall n \ge N$, $P(|Z_n - x| > \delta) \le \frac{\varepsilon}{2M}$. Hence

$$E[h(Z_n)] \le \frac{\varepsilon}{2} P(|Z_n - x| \le \delta) + MP(|Z_n - x| > \delta) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon.$$

It follows that $E[h(Z_n)] \to 0$, thus $E[f(Z_n)] \to f(x)$ as $n \to \infty$.)

On the other hand,

$$\begin{aligned} E[f(Z_n)] &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P(X_1 + \cdots + X_n = k) \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = B_n(x) \end{aligned}$$

Hence

$$\lim_{n \to \infty} B_n(x) = f(x).$$

p.415 Q10

For $i < \lambda$,

$$\begin{aligned} P(X \le i) &= \sum_{n=0}^{i} \frac{e^{-\lambda} \lambda^n}{n!} \\ &= \frac{e^{-\lambda} \lambda^i}{i^i} \sum_{n=0}^{i} \frac{i^n}{n!} \left(\frac{i}{\lambda}\right)^{i-n} \\ &\le \frac{e^{-\lambda} \lambda^i}{i^i} \sum_{n=0}^{\infty} \frac{i^n}{n!} \\ &= \frac{e^{i-\lambda} \lambda^i}{i^i}. \end{aligned}$$

Alternatively, one may use the Chernoff bound to obtain

$$P(X \le i) = P\left(e^{tX} \ge e^{ti}\right) \le e^{-ti} M_X(t) = e^{-ti} e^{\lambda\left(e^t - 1\right)}, \quad t < 0.$$

Putting $t = \log(i/\lambda)$, we have

$$P(X \le i) \le \left(\frac{\lambda}{i}\right)^i e^{i-\lambda}.$$