The Chinese University of Hong Kong

Department of Mathematics

MATH3280A Introductory Probability

Solutions to Assignment 6

p.373 Q4

(a)

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y)\ dx\ dy = \int_0^1 \int_0^y x\ dx\ dy = \frac{1}{6}.$$

(b)

$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x,y)\ dx\ dy = \int_0^1 \int_0^y x\frac{1}{y}\ dx\ dy = \frac{1}{4}.$$

(c)

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x,y)\ dx\ dy = \int_0^1 \int_0^y y\frac{1}{y}\ dx\ dy = \frac{1}{2}.$$

p.373 Q13

For $i = 1, 2, \ldots, 1000$, let X_i be the random variable such that $X_i = 1$ if the i-th person gets a card which matches his age, and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{1000} X_i$ is the number of matches. Since for each i, only one of the 1000 cards matches the age of the i-th person, we have

$$E(X_i) = P(X_i = 1) = 1/1000,$$

and it follows that

$$E(X) = \sum_{i=1}^{1000} E(X_i) = 1.$$

p.374 Q21

(a) For $i = 1, 2, \ldots, 365$, let X_i be the random variable such that $X_i = 1$ if the i-th day is a birthday of exactly three people, and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$ is the number of days that are birthdays of exactly three people. Note that for each i,

$$E(X_i) = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

Hence

$$E(X) = \sum_{i=1}^{365} E(X_i) = 365 \cdot \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97} \approx 0.9301.$$

(b) For $i = 1, 2, \ldots, 365$, let Y_i be the random variable such that $Y_i = 1$ if the i-th day is the birthday of at least one person, and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$ is the number of days that are birthdays of at least one person. Note that for each i,

$$E(Y_i) = P(Y_i = 1) = 1 - \left(\frac{364}{365}\right)^{100}.$$

Hence

$$E(Y) = \sum_{i=1}^{365} E(Y_i) = 365 \cdot \left(1 - \left(\frac{364}{365}\right)^{100}\right) \approx 87.5755.$$

p.375 Q30

Note that since X and Y are independent, we have $E(XY) = E(X)E(Y)$. Hence

$$\begin{aligned}
E((X-Y)^2) &= E(X^2 - 2XY + Y^2) \\
&= E(X^2) - 2E(XY) + E(Y^2) \\
&= Var(X) + E(X)^2 - 2E(X)E(Y) + Var(Y) + E(Y)^2 \\
&= \sigma^2 + \mu^2 - 2\mu^2 + \sigma^2 + \mu^2 \\
&= 2\sigma^2.
\end{aligned}$$

p.375 Q38

Note that $Cov(X,Y) = E(XY) - E(X)E(Y)$.

$$\begin{aligned}
E(XY) &= \int_0^\infty \int_0^x xy \cdot \frac{2e^{-2x}}{x}\, dy\, dx = \frac{1}{4}. \\
E(X) &= \int_0^\infty \int_0^x x \cdot \frac{2e^{-2x}}{x}\, dy\, dx = \frac{1}{2}. \\
E(Y) &= \int_0^\infty \int_0^x y \cdot \frac{2e^{-2x}}{x}\, dy\, dx = \frac{1}{4}.
\end{aligned}$$

Hence $Cov(X,Y) = 1/8$.

p.376 Q48

(a) We have

$$E(X) = \sum_{k=1}^\infty kP(X=k) = \sum_{k=1}^\infty k\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}.$$

Using the fact that for $|x| < 1$,

$$\sum_{k=1}^\infty kx^{k-1} = \frac{1}{(1-x)^2},$$

it follows that $E(X) = 6$.

(b)

$$
\begin{aligned}
E(X|Y=1) &= \sum_{k=1}^{\infty} k \cdot P(X=k|Y=1) \\
&= \sum_{k=2}^{\infty} k \left(\frac{5}{6}\right)^{k-2} \frac{1}{6} \\
&= \sum_{k=1}^{\infty} (1+k) \left(\frac{5}{6}\right)^{k-1} \frac{1}{6} \\
&= 1 + E(X) \\
&= 7.
\end{aligned}
$$

(c)

$$
\begin{aligned}
E(X|Y=5) &= \sum_{k=1}^{\infty} k \cdot P(X=k|Y=5) \\
&= \sum_{k=1}^{4} k \cdot P(X=k|Y=5) + \sum_{k=6}^{\infty} k \cdot P(X=k|Y=5)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{4} k \cdot P(X=k|Y=5) &= 1(1/5) + 2(4/5)(1/5) + 3(4/5)^2(1/5) + 4(4/5)^3(1/5) \\
&= \frac{821}{625} \\
\sum_{k=6}^{\infty} k \cdot P(X=k|Y=5) &= \sum_{k=6}^{\infty} k(4/5)^4(5/6)^{k-6}(1/6) \\
&= (4/5)^4 \sum_{k=1}^{\infty} (5+k)(5/6)^{k-1}(1/6) \\
&= (4/5)^4(5+E(X)) \\
&= \frac{2816}{625}.
\end{aligned}
$$

Hence

$$
E(X|Y=5) = \frac{3637}{625} \cdot
$$

p.376 Q50

The density of Y is

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} dx = e^{-y}, \quad y > 0,$$

and $f_Y(y) = 0$ if $y \leq 0$. Hence for $y > 0$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{e^{-y}},$$

and

$$E(X^2|Y = y) = \int_{-\infty}^\infty x^2 f_{X|Y}(x|y)\ dx = \int_0^\infty x^2 \frac{e^{-x/y}}{y}\ dx = 2y^2.$$

p.381 Q12

(a) Note that $E(X_n) = 1$ for each n. Since $\frac{X_n}{3^n} \geq 0$ for each n, by the monotone convergence theorem, we have

$$E(X) = \sum_{n=1}^\infty E(\frac{X_n}{3^n}) = \sum_{n=1}^\infty \frac{E(X_n)}{3^n} = \sum_{n=1}^\infty \frac{1}{3^n} = \frac{1}{2}.$$

(b) Note that $Var(X_n) = E(X_n^2) - E(X_n)^2 = 2 - 1^2 = 1$ for each n.
Let N be a positive integer. Since $X_1, X_2, \ldots, X_N$ are independent, we have

$$Var(\sum_{n=1}^N \frac{X_n}{3^n}) = \sum_{n=1}^N \frac{Var(X_n)}{3^{2n}} = \sum_{n=1}^N \frac{1}{9^n}. \tag{1}$$

Note that

$$Var(\sum_{n=1}^N \frac{X_n}{3^n}) = E\left((\sum_{n=1}^N \frac{X_n}{3^n} - \sum_{n=1}^N \frac{1}{3^n})^2\right)$$

converges to $E((\sum_{n=1}^\infty \frac{X_n}{3^n} - \sum_{n=1}^\infty \frac{1}{3^n})^2) = Var(\sum_{n=1}^\infty \frac{X_n}{3^n})$ as $N \to \infty$ by the dominated convergence theorem. Hence, taking limit $N \to \infty$ in (1), we have

$$Var(\sum_{n=1}^\infty \frac{X_n}{3^n}) = \sum_{n=1}^\infty \frac{1}{9^n} = \frac{1}{8}.$$

p.381 Q19

$$\begin{aligned} Cov(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\ &= E(X^2 - Y^2) - (E(X) + E(Y))(E(X) - E(Y)) \\ &= E(X^2) - E(Y^2) - (E(X)^2 - E(Y)^2). \end{aligned}$$

Since X and Y are identically distributed, it follows that $E(X) = E(Y)$ and $E(X^2) = E(Y^2)$. Hence $Cov(X+Y, X-Y) = 0$.