The Chinese University of Hong Kong
Department of Mathematics
MATH3280 Introductory Probability
Solutions to Assignment 5

p.287 Q2

(a) Let $p(x_1, x_2)$ be the joint probability mass X_1 and X_2.

$$p(0,0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39}$$
$$p(0,1) = \frac{8}{13} \cdot \frac{5}{12} = \frac{10}{39}$$
$$p(1,0) = \frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39}$$
$$p(1,1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39}$$

(b) Let $q(x_1, x_2, x_3)$ be the joint probability mass of X_1, X_2 and X_3.

$$q(0,0,0) = \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{28}{143} \qquad q(0,1,1) = \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} = \frac{40}{429}$$
$$q(0,0,1) = \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{5}{11} = \frac{70}{429} \qquad q(1,1,0) = \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{8}{11} = \frac{40}{429}$$
$$q(0,1,0) = \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{7}{11} = \frac{70}{429} \qquad q(1,0,1) = \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{4}{11} = \frac{40}{429}$$
$$q(1,0,0) = \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} = \frac{70}{429} \qquad q(1,1,1) = \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{3}{11} = \frac{5}{143}$$

p.287 Q9

(a) Since f is non-negative and

$$\iint_{\mathbb{R}^2} f(x,y)dxdy = \int_0^1 \int_0^2 \frac{6}{7} \left(x^2 + xy/2 \right) dydx = 1$$

it follows that f is a joint density function.

(b) The density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^2 \frac{6}{7} \left(x^2 + xy/2 \right) dy = \frac{6}{7} \left(2x^2 + x \right), \quad x \in (0,1)$$

and $f_X(x) = 0$ elsewhere.

(c)
$$P(X > Y) = \iint_{\{(x,y):x>y\}} f(x,y)dxdy = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + xy/2\right) dydx = \frac{15}{56}$$

(d)
$$\begin{aligned}
P(Y > 1/2 \mid X < 1/2) &= \frac{P(X < 1/2, Y > 1/2)}{P(X < 1/2)} \\
&= \frac{\int_0^{1/2} \int_{1/2}^2 \frac{6}{7}\left(x^2 + xy/2\right) dydx}{\int_0^{1/2} \frac{6}{7}\left(x^2 + xy/2\right) dx} \\
&= \frac{\frac{6}{7} \cdot \frac{23}{128}}{\frac{6}{7} \cdot \frac{5}{24}} \\
&= \frac{69}{80}
\end{aligned}$$

(e)
$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_0^1 x \cdot \frac{6}{7}\left(2x^2 + x\right) dx = \frac{5}{7}$$

(f) The density of Y is given by
$$f_Y(y) = \int_0^1 \frac{6}{7}\left(x^2 + xy/2\right) dx = \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right), \quad y \in (0, 2)$$

and $f_Y(y) = 0$ elsewhere. Thus
$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy = \int_0^2 y \cdot \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) dy = \frac{8}{7}.$$

p.287 Q10

(a)
$$P(X < Y) = \iint_{((z,y):x<y)} f(x,y)dxdy = \int_0^{\infty} \int_x^{\infty} e^{-(x+y)}dydx = \frac{1}{2}.$$

(b)
$$P(X < a) = \begin{cases} \int_0^a \int_0^{\infty} e^{-(x+y)}dydx = 1 - e^{-a} & a > 0 \\ 0 & a \le 0 \end{cases}$$

p.288 Q22

(a) The density of X and Y are given by
$$f_X(x) = \begin{cases} \int_0^1 (x+y)dy = x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

2

$$f_Y(y) = \begin{cases} \int_0^1 (x+y)dx = y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $f \ne f_X \cdot f_Y$, we conclude that X and Y are not independent.

(b) See $f_X(x)$ in (a).

(c)
$$P(X+Y<1) = \iint_{\{(z,y):x+y<1\}} f(x,y)dxdy = \int_0^1 \int_0^{1-x} (x+y)dydx = \frac{1}{3}$$

p.289 Q26

(a) The joint density of A, B and C is given by $f(a,b,c) = f_A(a) \cdot f_B(b) \cdot f_C(c)$. Thus the joint cumulative distribution of A, B and C is

$$F(a,b,c) = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^e f_A(a) \cdot f_B(b) \cdot f_C(c)dcdbda = F_A(a) \cdot F_B(b) \cdot F_C(c)$$

where
$$F_A(t) = F_B(t) = F_C(t) = \begin{cases} 1, & t \ge 1 \\ t, & 0 < t < 1 \\ 0, & t \le 0 \end{cases}$$

(b) Note that all roots of $Ax^2 + Bx + C$ are real if and only if $B^2 \ge 4AC$.

$$\begin{aligned}
P\left(B^2 \ge 4AC\right) &= \iiint_{\{(a,b,c)\in[0,1]^3:b^2\ge 4ac\}} f(a,b,c)da\,db\,dc \\
&= \int_0^{1/4} \int_0^1 \int_{\sqrt{4ac}}^1 db\,dc\,da + \int_{1/4}^1 \int_0^1 \int_0^{\frac{1}{4a}b^2} dc\,db\,da \\
&= \frac{5}{36} + \frac{1}{6} \ln 2
\end{aligned}$$

where the second equality is derived by the following argument:

- If $0 \le a \le 1/4$, then $4ac \le 1$ always hold for $0 \le c \le 1$, thus $\sqrt{4ac} \le b \le 1$.
- If $1/4 \le a \le 1$, then $b^2/4a \le 1$ always hold for $0 \le b \le 1$, thus $0 \le c \le b^2/4a$.

Alternatively,

$$\begin{aligned}
P\left(B^2 \ge 4AC\right) &= 1 - \iiint_{\{(a,b,c)\in[0,1]^3:b^2\le 4ac\}} f(a,b,c)da\,db\,dc \\
&= 1 - \int_0^1 \int_{b^2/4}^1 \int_{b^2/4a}^1 dc\,da\,db \\
&= \frac{5}{36} + \frac{1}{6} \ln 2.
\end{aligned}$$

3

p.290 Q56

(a) Let $g_1(x, y) = x + y$ and $g_2(x, y) = x/y$. Then

$$|J(x, y)| = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = \begin{vmatrix} \frac{x+y}{y^2} \end{vmatrix}$$

Note that

$$\begin{cases} U = X + Y \\ V = \frac{X}{Y} \end{cases} \Leftrightarrow \begin{cases} X = \frac{UV}{V+1} \\ Y = \frac{U}{V+1} \end{cases}$$

Hence the joint density of U and V is

$$f_{U,V}(u, v) = f(x, y) \cdot |J(x, y)|^{-1}$$
$$= f\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \cdot \left|J\left(\frac{uv}{v+1}, \frac{u}{v+1}\right)\right|^{-1}$$
$$= \begin{cases} \frac{u}{(v+1)^2} & 0 < \frac{u}{v+1} < 1, 0 < \frac{uv}{v+1} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let $g_1(x, y) = x$ and $g_2(x, y) = x/y$. Then

$$|J(x, y)| = \begin{vmatrix} 1 & 0 \\ 1/y & -x/y^2 \end{vmatrix} = \begin{vmatrix} \frac{x}{y^2} \end{vmatrix}$$

Note that

$$\begin{cases} U = X \\ V = \frac{X}{Y} \end{cases} \Leftrightarrow \begin{cases} X = U \\ Y = \frac{U}{V} \end{cases}$$

Hence the joint density of U and V is

$$f_{U,V}(u, v) = f(x, y) \cdot |J(x, y)|^{-1}$$
$$= f\left(u, \frac{u}{v}\right) \cdot \left|J\left(u, \frac{u}{v}\right)\right|^{-1}$$
$$= \begin{cases} \frac{u}{v^2} & 0 < u < 1, 0 < \frac{u}{v} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $g_1(x, y) = x + y$ and $g_2(x, y) = \frac{x}{x+y}$. Then

$$|J(x, y)| = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x+y} \end{vmatrix}$$

Note that

$$\begin{cases} U = X + Y \\ V = \frac{X}{X+Y} \end{cases} \Leftrightarrow \begin{cases} X = UV \\ Y = U - UV \end{cases}$$

4

Hence the joint density of U and V is

$$
\begin{aligned}
f_{U,V}(u,v) &= f(x,y) \cdot |J(x,y)|^{-1} \\
&= f(uv, u - uv) \cdot |J(u, u - uv)|^{-1} \\
&= \begin{cases} u & 0 < uv < 1, 0 < u - uv < 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

p.291 Q5

Let $f(x), g(y)$ be the densities of X, Y.

(a) Since the cumulative distribution of Z is

$$
F_Z(z) = P(X/Y \le z) = P(X \le Yz) = \int_0^\infty \int_0^{yz} f(x)g(y)dxdy,
$$

then the density of Z is

$$
h(z) = \frac{dF_Z(z)}{dz} = \int_0^\infty \frac{d}{dz} \int_0^{yz} f(x)g(y)dxdy = \int_0^\infty yf(yz)g(y)dy.
$$

When $z \le 0$, $h(z) = 0$.

(b) Since the cumulative distribution of Z is

$$
F_Z(z) = P(XY \le z) = P(X \le z/Y) = \int_0^\infty \int_0^{z/y} f(x)g(y)dxdy,
$$

then the density of Z is

$$
h(z) = \frac{dF_Z(z)}{dz} = \int_0^\infty \frac{d}{dz} \int_0^{z/y} f(x)g(y)dxdy = \int_0^\infty \frac{1}{y}f(\frac{z}{y})g(y)dy.
$$

When $z \le 0$, $h(z) = 0$.

If $f(x) = \lambda \exp(-\lambda x), x > 0$ and $g(y) = \eta \exp(-\eta y), y > 0$ for some $\lambda, \eta > 0$, then

(a)
$$
h(z) = \begin{cases} \lambda\eta \int_0^\infty ye^{-(\lambda z + \eta)y}dy = \dfrac{\lambda\eta}{(z\lambda + \eta)^2} & z > 0 \\ 0 & \text{otherwise.} \end{cases}
$$

(b)
$$
h(z) = \begin{cases} \lambda\eta \int_0^\infty \frac{1}{y}e^{-(\lambda z/y + \eta y)}dy & z > 0 \\ 0 & \text{otherwise.} \end{cases}
$$

5

p.292 Q17

$$P(X = n, Y = m) = P\left(X_1 + X_2 = n, X_2 + x_3 = m\right)$$

$$= \sum_{k=0}^{\min\{n,m\}} P\left(X_1 = n - k, X_2 = k, X_3 = m - k\right)$$

$$= \sum_{k=0}^{\min\{n,m]} P\left(X_1 = n - k\right) P\left(X_2 = k\right) P\left(X_3 = m - k\right)$$

$$= \sum_{k=0}^{\min\{n,m\}} \frac{e^{-\lambda_1}\lambda_1^{n-k}}{(n-k)!} \cdot \frac{e^{-\lambda_2}\lambda_2^{k}}{k!} \cdot \frac{e^{-\lambda_3}\lambda_3^{m-k}}{(m-k)!}$$

$$= e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{k=0}^{\min\{n,m)} \frac{\lambda_1^{n-k}\lambda_2^{k}\lambda_3^{m-k}}{(n-k)! \cdot k! \cdot (m-k)!} \cdot$$