The Chinese University of Hong Kong
Department of Mathematics

MATH3280A Introductory Probability

Solutions to Assignment 4

p.224 Q4

(a)

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\, dx = \frac{1}{2}.$$

(b) The cumulative distribution function of X is

$$F(t) = \begin{cases} \int_{10}^{t} \frac{10}{x^2}\, dx & t \geq 10 \\ 0 & t < 10 \end{cases}$$
$$= \begin{cases} 1 - \frac{10}{t} & t \geq 10 \\ 0 & t < 10 \end{cases}$$

(c) Assume that the lifetimes of the electronic devices are independent. Let Y be the random variable of the number of devices that will function for at least 15 hours. Then Y has a binomial distribution with parameters $n = 6$ and p, where

$$p = P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2}\, dx = \frac{2}{3}.$$

The required probability is

$$P(Y \geq 3) = 1 - \sum_{k=0}^{2} \binom{6}{k} p^k (1-p)^{6-k} = \frac{656}{729} \approx 0.8999$$

p.225 Q7

First, note that

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 (a + bx^2)\, dx = a + \frac{1}{3}b. \tag{1}$$

Moreover, we have

$$\frac{3}{5} = E(X) = \int_0^1 x(a + bx^2)\, dx = \frac{1}{2}a + \frac{1}{4}b. \tag{2}$$

Solving (1) and (2), we have $a = 3/5$ and $b = 6/5$.

p.225 Q11

We may identify the line segment with the interval $[0, L]$. A point x is randomly chosen on the interval and let X be the random variable given by the value of x. Then X is the uniform random variable over $[0, L]$. Note that

$$\left(X < L - X \text{ and } \frac{X}{L - X} < \frac{1}{4}\right) \Leftrightarrow X < \frac{L}{5}$$

and

$$\left(X > L - X \text{ and } \frac{L - X}{X} < \frac{1}{4}\right) \Leftrightarrow X > \frac{4L}{5}.$$

The required probability is

$$P(X < L/5) + P(X > 4L/5) = 1/5 + 1/5 = 2/5.$$

p.225 Q16

Assume that the annual rainfalls are independent from year to year. Let X be the random variable of annual rainfall. Then $X \sim N(40, 4^2)$.

$$P(X \leq 50) = P(\frac{X - 40}{4} \leq 2.5) = \Phi(2.5) \approx 0.9938$$

The required probability is $P(X \leq 50)^{10} \approx 0.9397$.

p.225 Q18

Suppose $X \sim N(5, \sigma^2)$. Then we have

$$0.2 = P(X > 9) = P(\frac{X - 5}{\sigma} > \frac{4}{\sigma}) = 1 - \Phi(4/\sigma)$$

i.e. $\Phi(4/\sigma) = 0.8$. Checking the standard normal table, we have $\Phi(0.84) \approx 0.8$ and thus $4/\sigma \approx 0.84$. Hence $Var(X) = \sigma^2 \approx 22.7$.

p.226 Q32

(a)

$$P(X > 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} \, dx = e^{-1}.$$

(b)

$$P(X \geq 10 | X > 9) = \frac{P(X \geq 10)}{P(X > 9)} = \frac{\int_{10}^\infty \frac{1}{2} e^{-\frac{1}{2}x} \, dx}{\int_9^\infty \frac{1}{2} e^{-\frac{1}{2}x} \, dx} = \frac{e^{-10/2}}{e^{-9/2}} = e^{-1/2}.$$

p.227 Q8

We assume that X is a continuous random variable with density $f(x)$.

$$\begin{aligned} E(X^2) &= \int_0^c x^2 f(x)\ dx \le c\int_0^c xf(x)\ dx = cE(X). \\ Var(X) &= E(X^2) - E(X)^2 \\ &\le cE(X) - E(X)^2 \\ &= -\left(E(X) - \frac{c}{2}\right)^2 + \frac{c^2}{4} \\ &\le \frac{c^2}{4}. \end{aligned}$$

p.228 Q11

(a) By integration by parts, we have

$$\begin{aligned} E(g'(Z)) &= \int_{-\infty}^{\infty} g'(x)\cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\ dx \\ &= \frac{1}{\sqrt{2\pi}}\left[g(x)e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} xg(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\ dx \end{aligned}$$

Here, we need the additional assumption that

$$\lim_{x\to\pm\infty} g(x)e^{-\frac{x^2}{2}} = 0.$$

Then we have $E(g'(Z)) = E(Zg(Z))$.

(b) For $n = 1$, $E(Z^2) = Var(Z) + E(Z)^2 = 1 = 1\cdot E(Z^0)$.
For $n \ge 2$, put $g(x) = x^n$. Note that the additional assumption in (a) is satisfied. Then by (a), we have $E(Z^{n+1}) = E(Zg(Z)) = E(g'(Z)) = E(nZ^{n-1}) = nE(Z^{n-1})$.

(c) By (b), we have $E(Z^4) = 3E(Z^2) = 3$.

p.228 Q15

Let F_X and F_{cX} be the distribution of X and cX respectively. Let f_X and f_{cX} be the density of X and cX respectively. For $t > 0$,

$$F_{cX}(t) = P(cX \le t) = P(X \le t/c) = F_X(t/c),$$

$$f_{cX}(t) = F'_{cX}(t) = \frac{1}{c}f_X(t/c) = \frac{\lambda}{c}e^{-\frac{\lambda}{c}t}.$$

For $t < 0$, $F_{cX}(t) = 0$ and $f_{cX}(t) = 0$. Hence cX is exponential with parameter λ/c.