The Chinese University of Hong Kong
Department of Mathematics

MATH3280A Introductory Probability

Solutions to Assignment 3

p.172 Q1

The image of X is $\{4, 2, 1, 0, -1, -2\}$.

$$P(X=4) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$
$$P(X=2) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$
$$P(X=1) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}$$
$$P(X=0) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$
$$P(X=-1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$
$$P(X=-2) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{4}{13}$$

p.172 Q3

The image of X is $\{3, 4, \ldots, 18\}$.

$$P(X=3) = P(X=18) = \frac{1}{216}$$
$$P(X=4) = P(X=17) = \frac{3}{216}$$
$$P(X=5) = P(X=16) = \frac{6}{216}$$
$$P(X=6) = P(X=15) = \frac{10}{216}$$
$$P(X=7) = P(X=14) = \frac{15}{216}$$
$$P(X=8) = P(X=13) = \frac{21}{216}$$
$$P(X=9) = P(X=12) = \frac{25}{216}$$
$$P(X=10) = P(X=11) = \frac{27}{216}$$

p.174 Q21

(a) $E(X)$ should be larger than $E(Y)$ because Y is evenly distributed for all its values while X is more weighted for larger values.

(b)

$$\begin{aligned} E(X) &= 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} \approx 39.28 \\ E(Y) &= 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37 \end{aligned}$$

p.175 Q25

$$\begin{aligned} P(X=1) &= 0.6 \times (1-0.7) + (1-0.6) \times 0.7 = 0.46 \\ E(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\ &= 1 \cdot 0.46 + 2 \cdot (0.6 \times 0.7) \\ &= 1.3 \end{aligned}$$

p.176 Q37

$$\begin{aligned} E(X^2) &= 40^2 \cdot \frac{40}{148} + 33^2 \cdot \frac{33}{148} + 25^2 \cdot \frac{25}{148} + 50^2 \cdot \frac{50}{148} \\ Var(X) &= E(X^2) - E(X)^2 \approx 82.20 \\ E(Y^2) &= 40^2 \cdot \frac{1}{4} + 33^2 \cdot \frac{1}{4} + 25^2 \cdot \frac{1}{4} + 50^2 \cdot \frac{1}{4} \\ Var(Y) &= E(Y^2) - E(Y)^2 = 84.5 \end{aligned}$$

p.180 Q3

Let $Y = \alpha X + \beta$ and F_Y be the distribution function of Y.

$$\begin{aligned} F_Y(t) &= P(\alpha X + \beta \leq t) \\ &= \begin{cases} P(X \leq \frac{t-\beta}{\alpha}) & \alpha > 0 \\ P(X \geq \frac{t-\beta}{\alpha}) & \alpha < 0 \end{cases} \\ &= \begin{cases} P(X \leq \frac{t-\beta}{\alpha}) & \alpha > 0 \\ 1 - P(X < \frac{t-\beta}{\alpha}) & \alpha < 0 \end{cases} \\ &= \begin{cases} F(\frac{t-\beta}{\alpha}) & \alpha > 0 \\ 1 - \lim_{s \to (\frac{t-\beta}{\alpha})^-} F(s) & \alpha < 0, \end{cases} \end{aligned}$$

where F is the distribution function of X.

p.180 Q10

$$
\begin{aligned}
E(\frac{1}{X+1}) &= \sum_{k=0}^{n} \frac{1}{k+1} P(X=k) \\
&= \sum_{k=0}^{n} \frac{1}{k+1} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \left(\sum_{i=0}^{n+1} \frac{(n+1)!}{i!(n+1-i)!} p^i (1-p)^{n+1-i} - (1-p)^{n+1} \right) \\
&= \frac{1}{(n+1)p} \left((p+(1-p))^{n+1} - (1-p)^{n+1} \right) \\
&= \frac{1-(1-p)^{n+1}}{(n+1)p}
\end{aligned}
$$

p.181 Q18

Note that a Poisson r.v. has parameter $\lambda > 0$ and $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$
Fix a $k \in \{0, 1, 2, \ldots\}$ and let $f_k(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$.
Note that $f_0(\lambda) = e^{-\lambda}, \lambda > 0$, so no maximum is attained for $k = 0$.
If $k > 0$, then

$$
\begin{aligned}
f_k'(\lambda) &= \frac{e^{-\lambda}}{k!}(k\lambda^{k-1} - \lambda^k) \\
&\begin{cases} > 0 & \text{if } \lambda < k \\ = 0 & \text{if } \lambda = k \\ < 0 & \text{if } \lambda < k \end{cases}
\end{aligned}
$$

Hence $\lambda = k$ maximizes $P(X=k)$.

p.181 Q19

$$\begin{aligned}
E(X^n) &= \sum_{k=1}^{\infty} k^n \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} \\
&= \lambda \sum_{i=0}^{\infty} (i+1)^{n-1} \frac{i^k e^{-\lambda}}{i!} \\
&= \lambda E((X+1)^{n-1}).
\end{aligned}$$

$$\begin{aligned}
E(X) &= \lambda \\
E(X^2) &= \lambda E(X+1) = \lambda(E(X)+1) = \lambda^2 + \lambda \\
E(X^3) &= \lambda E((X+1)^2) \\
&= \lambda(E(X^2 + 2X + 1)) \\
&= \lambda(E(X^2) + 2E(X) + 1) \\
&= \lambda(\lambda^2 + \lambda + 2\lambda + 1) \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}$$