**The Chinese University of Hong Kong**
**Department of Mathematics**
MATH3280A Introductory Probability
Solutions to Assignment 2

**p.102 Q1**
Let $E$ be the event that at least one of the dice lands on 6.
Let $F$ be the event that the dice land on different numbers.
Since $P(E \cap F) = 10/36$ and $P(F) = 30/36$, the conditional probability of $E$ given $F$ is

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1}{3}.$$

**p.102 Q4**
Let $E$ be the event that at least one of the dice lands on 6.
Let $F_i$ be the event that the sum of the dice is $i$, where $i = 2, 3, \ldots, 12$.

$$P(E \cap F_i) = \begin{cases} 0 & i = 2, 3, \ldots, 6 \\ 2/36 & i = 7, 8, \ldots, 11 \\ 1/36 & i = 12. \end{cases}$$

$$P(F_i) = \begin{cases} 6/36 & i = 7 \\ 5/36 & i = 8 \\ 4/36 & i = 9 \\ 3/36 & i = 10 \\ 2/36 & i = 11 \\ 1/36 & i = 12. \end{cases}$$

Hence the conditional probability of $E$ given $F_i$ is

$$P(E|F_i) = \frac{P(E \cap F_i)}{P(F_i)} = \begin{cases} 0 & i = 2, 3, \ldots, 6 \\ 1/3 & i = 7 \\ 2/5 & i = 8 \\ 1/2 & i = 9 \\ 2/3 & i = 10 \\ 1 & i = 11, 12. \end{cases}$$

**p.102 Q11**

(a)

$$P(B|A_s) = \frac{P(B \cap A_s)}{P(A_s)} = \frac{\frac{1}{52} \times \frac{3}{51} + \frac{3}{52} \times \frac{1}{51}}{\frac{1}{52} \times 1 + \frac{51}{52} \times \frac{1}{51}} = \frac{1}{17}$$

or

$$P(B|A_s) = \frac{\binom{3}{1}/\binom{52}{2}}{\binom{51}{1}/\binom{52}{2}} = \frac{1}{17}.$$

(b)

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{4}{52} \times \frac{3}{51}}{1 - \frac{48}{52} \times \frac{47}{51}} = \frac{1}{33}$$

or

$$P(B|A) = \frac{\binom{4}{2}/\binom{52}{2}}{1 - \left[\binom{48}{2}/\binom{52}{2}\right]} = \frac{1}{33}.$$

**p.103 Q 23**

Let $W$ be the event that white ball is drawn from urn II.
Let $T$ be the event that the transferred ball is white.

(a) By the law of total probability,

$$P(W) = P(T)P(W|T) + P(T^c)P(W|T^c) = \frac{2}{6} \times \frac{2}{3} + \frac{4}{6} \times \frac{1}{3} = \frac{4}{9}.$$

(b) By the Bayes' formula,

$$P(T|W) = \frac{P(T)P(W|T)}{P(W)} = \frac{\frac{2}{6} \times \frac{2}{3}}{\frac{4}{9}} = \frac{1}{2}.$$

**p.109 Q73**

(a) The probability that all children are of the same sex is equal to

$$\frac{1}{2^5} + \frac{1}{2^5} = \frac{1}{16}.$$

(b) The probability that the 3 eldest children are boys and the others are girls is equal to

$$\frac{1}{2^5} = \frac{1}{32}.$$

(c) The probability that exactly 3 children are boys is equal to

$$\frac{1}{2^5} \times \binom{5}{3} = \frac{5}{16}.$$

(d) The probability that the 2 eldest children are girls is equal to

$$1 \times 1 \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

(e) The probability that there is at least one girl is equal to

$$1 - \frac{1}{2^5} = \frac{31}{32}.$$

**p.110 Q1**

Note that

$$P(AB|A) = \frac{P(AB \cap A)}{P(A)} = \frac{P(AB)}{P(A)}$$

and

$$P(AB|A \cup B) = \frac{AB \cap (A \cup B))}{P(A \cup B)} = \frac{P(AB)}{P(A \cup B)}.$$

Since $A \subseteq A \cup B$ implies that $P(A) \leq P(A \cup B)$, it follows that $P(AB|A) \geq P(AB|A \cup B)$.

**p.112 Q16**

Let $E_n$ be the event that there are even number of successes among the first $n$ trials.
Let $S_n$ be the event that the $n$-th trial is a success. Then for $n \geq 1$,

$$\begin{aligned} P_n = P(E_n) &= P(S_n|E_{n-1}^c)P(E_{n-1}^c) + P(S_n^c|E_{n-1})P(E_{n-1}) \\ &= P(S_n)P(E_{n-1}^c) + P(S_n^c)P(E_{n-1}) \qquad \text{(the trials are independent)} \\ &= p(1 - P_{n-1}) + (1-p)P_{n-1}. \end{aligned}$$

Next, we prove that for all $n \in \mathbb{N}$,

$$P_n = \frac{1 + (1-2p)^n}{2}. \qquad (\star)$$

For $n = 1$, we have $P_1 = 1 - p = \frac{1+(1-2p)}{2}$.
Suppose $(\star)$ holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} P_{n+1} &= p(1 - P_n) + (1-p)P_n \\ &= p(1 - \frac{1 + (1-2p)^n}{2}) + (1-p)\frac{1 + (1-2p)^n}{2} \\ &= \frac{2p - p + p(1-2p)^n + 1 - p + (1-p)(1-2p)^n}{2} \\ &= \frac{1 + (1-2p)^{n+1}}{2}. \end{aligned}$$

Hence the result follows from Mathematical induction.

**p.113 Q25**

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\begin{aligned}
P(E|FG)P(G|F) + P(E|FG^c)P(G^c|F) &= \frac{P(EFG)}{P(FG)}\frac{P(FG)}{P(F)} + \frac{P(EFG^c)}{P(FG^c)}\frac{P(FG^c)}{P(F)} \\
&= \frac{P(EFG) + P(EFG^c)}{P(F)} \\
&= \frac{P(EF)}{P(F)} \\
&= P(E|F).
\end{aligned}
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