Math 3280 20-11-26

Review

- Conditional expectation $E[X|Y=y]$
- Calculate expectation by conditioning

$$E[X] = E[E[X|Y]]$$

§ 7.7 Moment generating functions.

Def. Let X be a r.v. and $t \in \mathbb{R}$. Define

$$M_X(t) = E[e^{tX}].$$

For convenience, we also write $M(t) = M_X(t)$ and call it the moment generating function of X.

Remark:

① $$e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \cdot X^n.$$

Hence

(1) $$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[X^n]$$

$\to$ n-th moment of X.

② If $M_X(t)$ exists and is finite for all $-t_0 < t < t_0$ for some $t_0 > 0$,

then

(2) $$E[X^n] = M_X^{(n)}(0), \quad n = 1, 2, \cdots$$

Example 1. Let X be a binomial r.v. with parameters (n, p).

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} P\{X=k\}$$

$$= \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

By Binomial Thm

$$= \left(e^t p + (1-p)\right)^n.$$

Example 2. Let X be a Poisson r.v. with parameter λ.

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{(e^t \cdot \lambda)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{(e^t \lambda)}$$

$$= e^{\lambda(e^t - 1)}.$$

Example 3. Let Z be a standard normal r.v.

$$M_Z(t) = E[e^{tZ}]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} \, e^{-\frac{z^2}{2}} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} \cdot e^{-\frac{(z-t)^2}{2}} \, dz$$

Letting $x = z - t$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} \, e^{-\frac{x^2}{2}} \, dx$$

$$= e^{t^2/2}$$

Example 4. Let X be a normal r.v. with mean μ and variance σ^2.

Notice that $Z := \dfrac{X-\mu}{\sigma}$ is a standard normal r.v.

Hence $X = \mu + \sigma Z$.

$$
\begin{aligned}
M_X(t) &= E\left[e^{t(\mu+\sigma Z)}\right] \\
&= E\left[e^{t\mu} \cdot e^{t\sigma Z}\right] \\
&= e^{t\mu} E\left[e^{t\sigma Z}\right] \\
&= e^{t\mu} \cdot M_Z(t\sigma) \\
&= e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} = e^{\frac{\sigma^2 t^2}{2} + \mu \cdot t}
\end{aligned}
$$

Thm 5. Let X, Y be two r.v.'s.

If $\exists\ t_0 > 0$ such that

$$M_X(t) = M_Y(t) \quad \text{for} \quad t \in (-t_0, t_0)$$

and both are finite.

Then X and Y have the same distribution

(i.e $F_X = F_Y$)

Due to this result, we say that the moment generating function determines the distribution.

Prop 6. If X and Y are independent then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

Pf. $M_{X+Y}(t) = E[e^{tX+tY}]$

$$= E[e^{tX} \cdot e^{tY}]$$

$$= E[e^{tX}] \cdot E[e^{tY}] \quad \text{(here we use the independency of } X \text{ and } Y\text{)}$$

$$= M_X(t) \cdot M_Y(t).$$

$\square$

Chap. 8 Limiting Thms.

§ 8.1

Q: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent, identically distributed (IID) r.v.'s.

What can one say about the limiting behavior of

$$\frac{X_1 + \cdots + X_n}{n}$$

as $n \to \infty$?

§8.2. Two basic inequalities.

Prop. 7 (Markov inequality)

Let X be a non-negative r.v. Then for any $a>0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}.$$

Pf. Define a new r.v. I such that

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a. \end{cases}$$

Since $X \geq 0$, we have

$$I \leq \frac{X}{a}$$

Hence $E[I] \leq E\left[\frac{X}{a}\right] = E[X]/a.$

But $E[I] = 1 \cdot P\{I=1\} + 0\, P\{I=0\}$

$$= P\{I=1\}$$

$$= P\{X \geq a\}.$$

Hence $$P\{X \geq a\} \leq E[X]/a. \quad \blacksquare$$

Prop 8 (Chebyshev's inequality)

Let X be a r.v. with finite mean μ and variance σ^2. Then $\forall\ \varepsilon > 0$,

$$P\{\ |X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}.$$

Pf. Let $Y = |X-\mu|^2$. Applying Markov inequality to Y, we have

$$P\{|X-\mu| \geq \varepsilon\} = P\{Y \geq \varepsilon^2\}$$

$$\leq \frac{E[Y]}{\varepsilon^2} = \frac{E[|X-\mu|^2]}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}.$$

$\blacksquare$

Example. 9.

Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Solution: Let X denote the number of items produced during a week.

Then X is a non-negative r.v.

By our assumption, $E[X] = 50$.

(a) Hence by Markov inequality

$$P\{X > 75\} \leq \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}.$$

$$P\{X>75\} = P\{X \geq 76\} \leq \frac{E[X]}{76} = \frac{50}{76}.$$

(b) $Var(X) = 25.$

$$P\{40 \le X \le 60\}$$

$$= P\{ |X-50| \le 10 \}$$

$$= 1 - P\{ |X-50| > 10 \}$$

Using the Chebyshev inequality, we have

$$P\{|X-50| > 10\} \le \frac{Var(X)}{10^2} = \frac{25}{100} = \frac{1}{4}$$

Hence $P\{40 \le X \le 60\} \ge 1 - \frac{1}{4} = \frac{3}{4} = 0.75.$