Math 3280 11-20-23

Review.

- $\mathrm{Cov}(X, Y) = E[(X-E[X])(Y-E[Y])]$

$$= E[XY] - E[X]E[Y].$$

- It is bi-linear.
- $\mathrm{Cov}(X, Y) = 0$ if X, Y are independent.
- If $X_1, \cdots, X_n$ are independent, then

$$\mathrm{Var}(X_1 + \cdots + X_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k).$$

§ 7.5 Conditional expectations.

Def. If X and Y are discrete, then

the conditional expectation of X given $Y=y$, is

$$E[X \mid Y=y] := \sum_{x} x \, P\{X=x \mid Y=y\}$$

provided that $P\{Y=y\} > 0$.

Def. In the case when X and Y are jointly cts with a density $f(x,y)$, the conditional expectation of X given $Y=y$, is defined by

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx,$$

provided that $f_Y(y) > 0$, where

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Example 1. Let X, Y be jointly cts with a density

$$f(x,y) = \begin{cases} e^{-x/y} \cdot e^{-y}/y & \text{if } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $E[X \mid Y=y]$, $y>0$.

Solution: $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^{\infty} e^{-x/y} e^{-y}/y \; dx$$

$$= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty}$$

$$= e^{-y}, \quad \text{if} \quad y>0$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= e^{-x/y}/y \quad \text{if} \quad x, y > 0.$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$= \int_0^{\infty} x \cdot e^{-x/y}/y \; dx$$

$$\overset{\text{Int by Part}}{=} x \cdot (-e^{-x/y}) \Big|_{x=0}^{+\infty} + \int_0^{\infty} e^{-x/y} \cdot dx$$

$$= 0 + (-y e^{-x/y}) \Big|_{x=0}^{+\infty}$$

$$= y \quad \text{if} \quad y>0.$$

Now write

$E[X|Y]$ as a function of Y by

$$y \longmapsto E[X|Y=y]$$

$E[X|Y]$ is a r.v., the value of which depends on the value of Y.

Prop 2. $E[X] = E[\,E[X|Y]\,]$

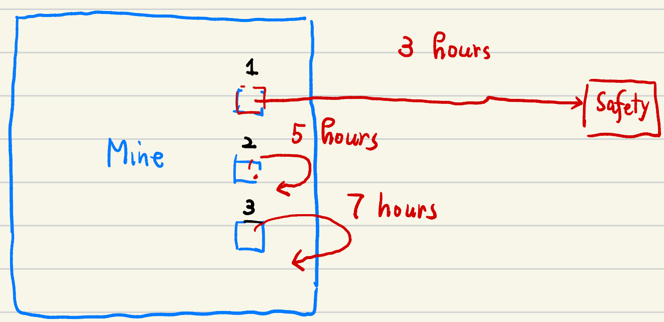
Pf. We only prove it in the discrete case.

$$
\begin{aligned}
E[\,E[X|Y]\,] &= \sum_y E[X|Y=y]\cdot P_Y(y) \\
&= \sum_y \cdot \sum_x x\cdot P\{X=x|Y=y\}\cdot P_Y(y) \\
&= \sum_y \sum_x x\, P\{X=x, Y=y\} \\
&= \sum_x \sum_y x\, P\{X=x, Y=y\} \\
&= \sum_x x\, P\{X=x\} = E[X]
\end{aligned}
$$

$\square$

Example 3.

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?



Solution: Let X denote the length of time (in hours) until the miner reaches safety.

Let Y denote the door that he choose in the first time.

By Prop 2,

$$E[X] = E[E[X|Y]]$$

$$= E[X|Y=1] \cdot P\{Y=1\}$$

$$+ E[X|Y=2] \cdot P\{Y=2\}$$

$$+ E[X \mid Y=3] \cdot P\{Y=3\}$$

$$= \frac{1}{3}\Big(E[X \mid Y=1] + E[X \mid Y=2] + E[X \mid Y=3]\Big)$$

$$= \frac{1}{3}\Big(3 + (5 + E[X]) + (7 + E[X])\Big)$$

Solving this equation, we obtain

$$E[X] = 3 + 5 + 7 = 15 \quad \text{(hours)}.$$

$\blacksquare$